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Intersection multiplicities and Grothendieck spaces

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Let $X, Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ with $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$. The intersection multiplicity of X and Y is defined as

$$\chi(X, Y) = \sum_i (-1)^i \operatorname{length} \mathrm{H}_i(X \otimes^{\mathbf{L}}_R Y)$$

whenever $X \in \mathsf{P}^{\mathrm{f}}(R)$ or $Y \in \mathsf{P}^{\mathrm{f}}(R)$.

Intersection multiplicities

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Theorem (Dutta–Hochster–McLaughlin)**.** *The vanishing conjecture does not hold, so not all rings satisfy vanishing!*

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$$\chi_\infty(X, Y) = \lim_{e \to \infty} \frac{1}{p^{e \operatorname{codim}(\operatorname{Supp} X)}} \chi(\mathbf{L}F^e(X), Y).$$

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$$[X] = 0 \qquad \text{whenever } X \text{ is exact}$$

$$[X] = [X'] + [X''] \qquad \text{whenever } 0 \to X' \to X \to X'' \to 0 \text{ is a short exact sequence in } \mathsf{C}.$$

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Let $\mathfrak{X}^c$ denote the maximal subset of $\operatorname{Spec} R$ such that

$$\mathfrak{X} \cap \mathfrak{X}^c = \{\mathfrak{m}\} \quad \text{and} \quad \dim \mathfrak{X} + \dim \mathfrak{X}^c \leq \dim R.$$

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Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. The Grothendieck space of $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ presented by generators $[X]$, one for each isomorphism class of a complex X in $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, and relations

$$[X] = [X'] \quad \text{when} \quad \chi(X, -) = \chi(X', -)$$

as metafunctions $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \to \mathbb{Q}$.

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The space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is equipped with the initial topology of the family of $\mathbb{Q}$-linear maps

$$\chi(-, Y) \colon \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{Q} \quad \text{for} \quad Y \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c).$$

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where $\mathfrak{X} = \operatorname{Supp} X$ and $F^e_{\mathfrak{X}}([X]) = [\mathbf{L}F^e(X)]$.

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where $\mathfrak{X} = \operatorname{Supp} X$ and $\lim_{e\to\infty} \Phi^e_{\mathfrak{X}}([X]) \in \mathbf{G}\mathrm{P}^{\mathrm{f}}(\mathfrak{X})$.

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$$\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$$

in which each $\alpha^{(i)}$ is either zero or an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue p^{-i}.

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Theorem 1. *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$ *and let $\alpha \in \mathrm{GP}^{\mathrm{f}}(\mathfrak{X})$. Then there is a unique decomposition*

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in which each $\alpha^{(i)}$ is either zero or an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue p^{-i}. The components $\alpha^{(0)}, \ldots, \alpha^{(u)}$ are recursively defined by

$$\alpha^{(0)} = \lim_{e \to \infty} \Phi_{\mathfrak{X}}^{e}(\alpha) \quad \textit{and}$$

$$\alpha^{(i)} = \lim_{e \to \infty} p^{ie} \Phi_{\mathfrak{X}}^{e}(\alpha - (\alpha^{(0)} + \cdots + \alpha^{(i-1)})),$$

Grothendieck spaces

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We have $u \leq \max(\operatorname{codim} \mathfrak{X} - 2, 0)$.

Grothendieck spaces

Translating the theorem to complexes, the Dutta multiplicity $\chi_\infty(X, Y)$ can be computed.

Grothendieck spaces

Translating the theorem to complexes, the Dutta multiplicity $\chi_\infty(X, Y)$ is the first entry in

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ p^t & p^{t-1} & \cdots & p^{t-u} \\ \vdots & \vdots & \ddots & \vdots \\ p^{ut} & p^{u(t-1)} & \cdots & p^{u(t-u)} \end{pmatrix}^{-1} \begin{pmatrix} \chi(X, Y) \\ \chi(\mathbf{L}F(X), Y) \\ \vdots \\ \chi(\mathbf{L}F^u(X), Y) \end{pmatrix},$$

where $t = \operatorname{codim}(\operatorname{Supp} X)$.

Numerical properties

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Theorem 2. *The ring R satisfies numerical vanishing if and only if all elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\{\mathfrak{m}\})$ do.*

Numerical properties

Assume that R is complete of prime characteristic p and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. If $\chi(\alpha, Y) = \chi(\alpha^{(0)}, Y)$ for all $Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$, then α satisfies numerical vanishing. If this holds for all elements in all Grothendieck spaces, then R satisfies numerical vanishing.

Theorem 2. *The ring R satisfies numerical vanishing if and only if all elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\{\mathfrak{m}\})$ do. In particular, this holds if and only if*

$$\chi(\mathbf{L}F(Z)) = p^{\dim R}\chi(Z)$$

for all complexes Z in $\mathsf{P}^{\mathrm{f}}(\{\mathfrak{m}\})$.

Numerical properties

The duality functor $\mathbf{R}\mathrm{Hom}_R(-, R)$ on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ induces an automorphism $(-)^*$ on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$.

Numerical properties

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An element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is self-dual if it satisfies $\alpha = (-1)^{\operatorname{codim}\mathfrak{X}}\alpha^*$.

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NUMERICAL PROPERTIES

The duality functor $\mathbf{R}\mathrm{Hom}_R(-,R)$ on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ induces an automorphism $(-)^*$ on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$.

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Theorem 3 (w. Frankild). *When R is complete of prime characteristic p and with perfect residue field,*

$$(-1)^{\operatorname{codim}\mathfrak{X}}\alpha^* = \alpha^{(0)} - \alpha^{(1)} + \cdots + (-1)^u\alpha^{(u)}.$$

Ring properties

vanishing

weak vanishing

Ring properties

$$\begin{array}{c} \text{vanishing} \\ \Downarrow \\ \text{weak vanishing} \end{array}$$

Ring properties

$$\begin{tikzcd}
\text{self-duality} \arrow[d, Leftrightarrow] \\
\text{vanishing} \arrow[d, Leftrightarrow] \\
\text{weak vanishing}
\end{tikzcd}$$

Ring properties

$$\begin{array}{c}
\text{self-duality} \\
\Updownarrow \\
\text{vanishing} \\
\Downarrow \\
\text{numerical self-duality} \\
\Downarrow \\
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\end{array}$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] \\
\text{regular} \arrow[r, Rightarrow] & \text{vanishing} \arrow[dd, Rightarrow] \\
 & \text{numerical self-duality} \arrow[d, Rightarrow] \\
 & \text{weak vanishing}
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] & \text{vanishing} \arrow[d, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
 & \begin{array}{c}\text{numerical}\\ \text{self-duality}\end{array} \arrow[d, Rightarrow] & \\
 & \text{weak vanishing} &
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[dd, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\begin{array}{c}\text{complete}\\ \text{intersection}\end{array} & & \\
 & \begin{array}{c}\text{numerical}\\ \text{self-duality}\end{array} \arrow[d, Rightarrow] & \\
 & \text{weak vanishing} &
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[dd, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\text{complete intersection} \arrow[dr, Rightarrow] & & \\
 & \text{numerical self-duality} \arrow[d, Rightarrow] & \\
 & \text{weak vanishing} &
\end{tikzcd}
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\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[dd, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\text{complete intersection} \arrow[dr, Rightarrow] & & \\
 & \text{numerical self-duality} \arrow[d, Rightarrow] & \\
 & \text{weak vanishing} & \dim \leq 4 \arrow[l, Rightarrow]
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[dd, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\text{complete intersection} \arrow[ddr, Rightarrow] & & \\
 & \text{numerical self-duality} \arrow[d, Rightarrow] & \text{Gorenstein of } \dim \leq 5 \arrow[dl, Rightarrow] \\
 & \text{weak vanishing} & \dim \leq 4 \arrow[l, Rightarrow]
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[dd, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\text{complete intersection} \arrow[dr, Rightarrow] & & \\
 & \text{numerical self-duality} \arrow[d, Rightarrow] & \text{Gorenstein of } \dim \leq 5 \arrow[dl, Rightarrow] \\
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\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[dd, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\text{complete intersection} \arrow[dr, Rightarrow] & & \\
 & \text{numerical self-duality} \arrow[d, Rightarrow] & \text{Gorenstein of } \dim \leq 5 \arrow[l, Rightarrow] \\
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\end{tikzcd}
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 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[d, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\text{complete intersection} \arrow[dr, Rightarrow] & \text{numerical vanishing} \arrow[d, Rightarrow] & \\
 & \text{numerical self-duality} \arrow[d, Rightarrow] & \text{Gorenstein of } \dim \leq 5 \arrow[l, Rightarrow] \\
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\text{complete intersection} \arrow[r, Rightarrow] & \text{numerical vanishing} \arrow[d, Rightarrow] & \\
 & \text{numerical self-duality} \arrow[d, Rightarrow] & \text{Gorenstein of } \dim \leq 5 \arrow[l, Rightarrow] \\
 & \text{weak vanishing} & \dim \leq 4 \arrow[l, Rightarrow]
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[d, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\begin{array}{c}\text{complete}\\ \text{intersection}\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\text{numerical}\\ \text{vanishing}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{Gorenstein}\\ \text{of dim} \leq 3\end{array} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
 & \begin{array}{c}\text{numerical}\\ \text{self-duality}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{Gorenstein}\\ \text{of dim} \leq 5\end{array} \arrow[l, Rightarrow] \\
 & \text{weak vanishing} & \dim \leq 4 \arrow[l, Rightarrow]
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[d, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\begin{array}{c}\text{complete}\\ \text{intersection}\end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{numerical}\\ \text{vanishing}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{Gorenstein}\\ \text{of dim} \leq 3\end{array} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\text{Gorenstein} & \begin{array}{c}\text{numerical}\\ \text{self-duality}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{Gorenstein}\\ \text{of dim} \leq 5\end{array} \arrow[l, Rightarrow] \\
 & \text{weak vanishing} & \dim \leq 4 \arrow[l, Rightarrow]
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[d, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\begin{array}{c}\text{complete}\\ \text{intersection}\end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{numerical}\\ \text{vanishing}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{Gorenstein}\\ \text{of dim} \leq 3\end{array} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\text{Gorenstein} \arrow[d, Rightarrow] & \begin{array}{c}\text{numerical}\\ \text{self-duality}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{Gorenstein}\\ \text{of dim} \leq 5\end{array} \arrow[l, Rightarrow] \\
\text{Cohen–Macaulay} & \text{weak vanishing} & \dim \leq 4 \arrow[l, Rightarrow]
\end{tikzcd}
$$

Ring properties

$$
\begin{tikzcd}
 & \text{self-duality} \arrow[d, Leftrightarrow] & \\
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[d, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\begin{array}{c}\text{complete}\\ \text{intersection}\end{array} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{numerical}\\ \text{vanishing}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{Gorenstein}\\ \text{of dim} \leq 3\end{array} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
\text{Gorenstein} \arrow[r, Rightarrow, dashed, "?"] \arrow[d, Rightarrow] & \begin{array}{c}\text{numerical}\\ \text{self-duality}\end{array} \arrow[d, Rightarrow] & \begin{array}{c}\text{Gorenstein}\\ \text{of dim} \leq 5\end{array} \arrow[l, Rightarrow] \\
\text{Cohen–Macaulay} & \text{weak vanishing} & \dim \leq 4 \arrow[l, Rightarrow]
\end{tikzcd}
$$