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Algebraic K-theory and local Chern characters

applied to Serre's conjectures on intersection multiplicity

Abstract

If M and N are finitely generated modules over a commutative, Noetherian, local ring R, such that M has finite projective dimension and $M \otimes_R N$ has finite length, then Serre's intersection multiplicity is defined as $\chi^R(M,N) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell \operatorname{length} \operatorname{Tor}_\ell^R(M,N)$. It has been conjectured that (0) $\dim_R M + \dim_R N \leq \dim R$; that (1) $\chi^R(M,N) = 0$ when this not an equality; and that (2) $\chi^R(M,N) > 0$ when it is. All three conjectures are proved in the case that $\dim_R N \leq 1$ by replacing M by a simpler module. This simplification is obtained by factoring $\chi^R(-,N)$ through a Grothendieck group and exploiting one among many surprising isomorphisms between these. Conjecture (1) is proved in the case where R is a complete intersection and both modules have finite projective dimension, using the properties of local Chern characters. The Grothendieck groups and local Chern characters turn out to be connected.

Contents

Preface

This text constitutes my master thesis for the cand. scient. degree in mathematics at the University of Copenhagen. It is the result of many months of work under the adept supervision of Hans-Bjørn Foxby.

Already during my first year as a mathematics student, I realized that my main interest was in algebra, but it was not until I was an exchange student at the University of California at Berkeley in the academic year 2001/2002 that I was able to narrow it down: it had to be commutative algebra, preferably with a flavor of category theory to it. Homological algebra satisfied these requirements perfectly, so before I returned to Copenhagen I asked Hans-Bjørn Foxby to be my adviser and he accepted. He then chose the subject of intersection multiplicities, and I accepted, eventually putting emphasis on the algebraic K-theory, as it satisfied my craving for category theory to an even greater extent.

The reader is assumed to be knowledgeable about rings, modules and complexes at a level corresponding to graduate courses in homological algebra. Nevertheless, to assist the reader, the preliminaries outline basic definitions, notation and terms together with a variety of fundamental results that are presented without proof. The remainder of the thesis is structured as follows.

Chapter 1 discusses the motivation behind the work in this thesis: Serre's conjectures on intersection multiplicity. Chapter 2 introduces the zeroth algebraic K-groups together with the more general concept of Grothendieck groups of complexes, and Chapter 3 establishes a multitude of surprising isomorphisms between various Grothendieck groups. Chapter 4 presents the first algebraic K-groups and connects them to the Grothendieck groups via the localization sequence; this leads to the introduction of an invariant known as the MacRae ideal. Chapter 5 changes the subject slightly by digressing into a superficial discussion of the theory of local Chern characters. Finally, Chapter 6 compiles the results established in previous chapters to prove some of Serre's conjectures in special cases and to discuss the application of algebraic K-theory and local Chern characters and the relationship between these in future research.

The core of the thesis is the work done in Chapters 2, 3 and 4. My main sources of information for the making of these chapters have been the book [Mag02] by Magurn and the paper [Fox82a] by Foxby, supplemented by the notes [Bas74] by Bass. Most of the theory in these chapters has been gathered from these sources, although the notation has been altered, theorems generalized and lemmas introduced to comply with the style of the thesis. In many cases I have provided more detailed proofs than were given in the sources, and at some points I have come up with proofs myself when none were at hand: this includes Theorem 3.23, which was only proven for $d = 1$ in [Fox82a]; the first part of Theorem 4.16, which was presented without proof in [Mag02] and proven somewhat differently in [Bas74] using a more general version of K_1-groups; and Proposition 4.5, which was presented without proof in [Mag02] and had a completely different proof in [Bas74].

Chapter 5 should only be seen as supplementing the discussion in Chapters 2, 3 and 4. It contains almost no proofs and does not even define the subject of the chapter: local Chern characters. The source for this chapter is Roberts' book [Rob98], supplemented by Foxby's notes [Fox89].

The box on the following page describes some basic assumptions that remain valid throughout the entire text. In addition to this, some sections are introduced with a box describing the additional assumptions made in that section alone. The reader is advised to pay careful attention to these boxes! *The content of a box remains valid throughout the section in which it is positioned.*

I am deeply indebted to my adviser, Hans-Bjørn Foxby, for taking the time to answer my numerous questions and for provoking me at the right times to come up with answers myself. I am also grateful to David Jeffrey Breuer for copyediting the text. Finally, I would like to thank my wonderful family, especially my mother, Lone Bo Halvorsen, and my father, Christian Henning Bistrup, for their unlimited love, support and complete faith in me and the projects I undertake.

Esben Bistrup Halvorsen
Copenhagen, January 2004

Throughout this thesis, R will denote a nontrivial, unitary, commutative ring. The unit element of R will be denoted by 1 and the zero element by 0. Unless otherwise stated, all ideals, modules and complexes are assumed to be ideals of R, R-modules and R-complexes, respectively.

Preliminaries

These preliminaries briefly summarize terms, notation and the basic results of modules, complexes and various constructions and invariants associated with these. Readers already familiar with the fundamental concepts of homological algebra will find nothing new here and should be able to quickly skim through the following pages.

Basic facts

Throughout this thesis, the symbol $\subseteq$ means "contained in or equal to", whereas the symbol $\subset$ is used when equality is not an option. The blackboard bold letters $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}_+$ and $\mathbb{Q}$ will denote the sets of positive integers, nonnegative integers, integers, positive rational numbers and rational numbers, respectively.

If $x = (x_1, \ldots, x_n)$ is a sequence of elements in a module, the submodule generated by these elements will be denoted by $\langle x_1, \ldots, x_n \rangle$ or simply $\langle x \rangle$. In particular, if $x = (x_1, \ldots, x_n)$ is a sequence in R, $\langle x \rangle$ denotes the ideal generated by the elements $x_1, \ldots, x_n$.

The multiplicative group of units of the ring R is denoted by R^*. A subset S of R is a *multiplicative system* if any product of elements from S is contained in S; this includes the "empty product" $1 \in R$. The complement of a prime ideal is a multiplicative system. There is a sort of inverse of this statement: any ideal that is maximal among ideals not intersecting a multiplicative system S is prime (cf. [Eis95, Proposition 2.11]).

When S is a multiplicative system in R and M is a module, the *localization of* M *at* S is denoted by $S^{-1}M$. If $S = R \backslash \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, then $S^{-1}M$ is referred to as the *localization of* M *at* $\mathfrak{p}$ and denoted by $M_\mathfrak{p}$. Given a multiplicative system S in R and a module M, there is an isomorphism $S^{-1}M \cong S^{-1}R \otimes_R M$, and since $S^{-1}R$ is itself a ring, this means that $S^{-1}M$ has the structure of an R-module as well as of an $S^{-1}R$-module. The module M is said to be S-*torsion* if $S^{-1}M = 0$. We extend these terms to the situation in which there are several multiplicative systems: if $S = (S_1, \ldots, S_d)$ is a family of multiplicative systems, M is said to be S-*torsion* if M is S_ν-torsion for $\nu = 1, \ldots, d$.

The set of prime ideals of R is denoted $\operatorname{Spec} R$ and referred to as the *spectrum* of R. If I is an ideal, we define $V_R(I)$ to be the set of prime ideals containing I. The intersection of all the primes in $V_R(I)$ is denoted by $\operatorname{Rad}_R I$ and referred to as the *radical* of I; it is the set of elements $r \in R$ such that $r^n \in I$ for some $n \in \mathbb{N}$ (cf. [Eis95, Corollary 2.12]). A prime ideal $\mathfrak{p}$ is *minimal over* I if it is minimal among elements in $V_R(I)$ ordered by inclusion. A prime ideal is simply called *minimal* if it is minimal over the zero ideal. When R is Noetherian, there are only finitely many minimal primes over an ideal I (cf. [Eis95, Exercise 1.2]).

A useful result about prime ideals is the *prime avoidance lemma* (cf. [Eis95, Lemma 3.3]):

Lemma 0.1 (prime avoidance). *If $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are ideals of which at most two are not prime, and if I is an ideal contained in $\cup_i \mathfrak{p}_i$, then I is contained in one of the $\mathfrak{p}_i$'s.* □

Suppose that M is a module. A *zerodivisor* on M is an element $r \in R$ such that $rm = 0$ for some nonzero $m \in M$. The set of zerodivisors on M is denoted $\operatorname{Zd}_R M$. The *annihilator* of M is the ideal $\operatorname{Ann}_R M = \{r \in R \,|\, rM = 0\}$, and the *support of* M is the set $\operatorname{Supp}_R M$ of prime ideals $\mathfrak{p}$ such that $M_\mathfrak{p}$ is nontrivial. If M is finitely generated and S is a multiplicative system, then M is S-torsion if and only if $\operatorname{Ann}_R M \cap S \neq \emptyset$; in particular, $\operatorname{Supp}_R M = V_R(\operatorname{Ann}_R M)$ (cf. [Eis95, Proposition 2.1 and Corollary 2.7]). If $m \in M$, the *annihilator* of m is the ideal $\operatorname{Ann}_R(m) = \{r \in R \mid rm = 0\}$. If the annihilator of an element of M turns out to be a prime ideal, this ideal is said to be *associated to* M. The set of associated primes of M is denoted by $\operatorname{Ass}_R M$. When R is Noetherian and M is finitely generated and nontrivial, $\operatorname{Ass}_R M$ is a nonempty, finite subset of $\operatorname{Supp}_R M$, including all the minimal primes over $\operatorname{Ann}_R M$, and the union of the primes in $\operatorname{Ass}_R M$ equals $\operatorname{Zd}_R M$ (cf. [Eis95, Theorem 3.1]). If R is Noetherian and $0 \to L \to M \to N \to 0$ is a short exact sequence of modules, then $\operatorname{Supp}_R M = \operatorname{Supp}_R L \cup \operatorname{Supp}_R N$ (cf. [Rob98, Proposition 1.1.3]). If R is Noetherian and local, and M and N are finitely generated modules, then $\operatorname{Supp}_R(M \otimes_R N) = \operatorname{Supp}_R M \cap \operatorname{Supp}_R N$ (cf. [Rob98, Proposition 2.3.2]).

Complexes

A *complex* X is a family $(X_\ell)_{\ell \in \mathbb{Z}}$ of modules together with a family $(\partial_\ell^X)_{\ell \in \mathbb{Z}}$ of homomorphisms $\partial_\ell^X \colon X_\ell \to X_{\ell-1}$, which are called the *differentials* of X, such that $\partial_\ell^X \partial_{\ell+1}^X = 0$ for all $\ell \in \mathbb{Z}$:

$$X = \quad \cdots \longrightarrow X_{\ell+1} \xrightarrow{\partial_{\ell+1}^X} X_\ell \xrightarrow{\partial_\ell^X} X_{\ell-1} \longrightarrow \cdots .$$

The complex X is said to be *bounded* if $X = 0$ for all but finitely many ℓ. More specifically, X is said to be *concentrated in degrees* $\ell_1, \dots, \ell_t$ if $X_\ell \neq 0$ implies

$\ell \in \{\ell_1, \ldots, \ell_t\}$. From now on, modules will always be thought of as complexes concentrated in degree 0: that is, complexes X with $X_\ell = 0$ for all $\ell \neq 0$.

The *zero complex* is the complex $\cdots \to 0 \to 0 \to 0 \to \cdots$ having the zero module in each degree. For $n \in \mathbb{Z}$, the complex X *shifted* n degrees to the left is the complex $\Sigma^n X$ given by $(\Sigma^n X)_\ell = X_{\ell-n}$ and $\partial_\ell^{\Sigma^n X} = (-1)^n \partial_{\ell-n}^X$ for all $\ell \in \mathbb{Z}$. The change of sign of the differentials turns out to be natural in connection with the mapping cone, which is described below. In the case that $n = 1$, the operator $\Sigma^1(-)$ is denoted simply by $\Sigma(-)$. The *direct sum* of two complexes X and Y is the complex $X \oplus Y$ given by $(X \oplus Y)_\ell = X_\ell \oplus Y_\ell$ and $\partial_\ell^{X \oplus Y} = \left(\begin{smallmatrix} \partial_\ell^X & 0 \\ 0 & \partial_\ell^Y \end{smallmatrix}\right)$ for all $\ell \in \mathbb{Z}$.

When X and Y are complexes, a *morphism* $\phi \colon X \to Y$ is a family $\phi = (\phi_\ell)_{\ell \in \mathbb{Z}}$ of homomorphisms $\phi_\ell \colon X_\ell \to Y_\ell$, making the following diagram commutative.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_{\ell+1} & \xrightarrow{\partial_{\ell+1}^X} & X_\ell & \xrightarrow{\partial_\ell^X} & X_{\ell-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \phi_{\ell+1}} & & \downarrow{\scriptstyle \phi_\ell} & & \downarrow{\scriptstyle \phi_{\ell-1}} & & \\ \cdots & \longrightarrow & Y_{\ell+1} & \xrightarrow{\partial_{\ell+1}^Y} & Y_\ell & \xrightarrow{\partial_\ell^Y} & Y_{\ell-1} & \longrightarrow & \cdots \end{array}$$

The *identity morphism* on the complex X is the morphism $\mathbb{1}_X \colon X \to X$ given for each $\ell \in \mathbb{Z}$ by $(\mathbb{1}_X)_\ell = \mathbb{1}_{X_\ell}$, where $\mathbb{1}_{X_\ell} \colon X_\ell \to X_\ell$ is the identity map. The *composition* of two morphisms $\phi \colon X \to Y$ and $\psi \colon Y \to Z$ is the morphism $\psi\phi \colon X \to Z$ given by $(\psi\phi)_\ell = \psi_\ell \phi_\ell$.

A morphism $\phi \colon X \to Y$ is an *isomorphism* if a morphism $\phi^{-1} \colon Y \to X$ exists such that $\phi^{-1}\phi = \mathbb{1}_X$ and $\phi\phi^{-1} = \mathbb{1}_Y$. In this case, we say that X is *isomorphic* to Y, and we write $\phi \colon X \xrightarrow{\cong} Y$ or simply $X \cong Y$. A necessary and sufficient condition for a morphism of complexes to be an isomorphism is that it is an isomorphism of modules in each degree. Note, however, that this does not mean that complexes with isomorphic modules are isomorphic!

A *short exact sequence of complexes* is a sequence $0 \to X \to Y \to Z \to 0$ of morphisms such that, for each $\ell \in \mathbb{Z}$, the sequence $0 \to X_\ell \to Y_\ell \to Z_\ell \to 0$ of modules in degree ℓ is exact.

A *homotopy* between two morphisms $\phi, \psi \colon X \to Y$ is a family $h = (h_\ell)_{\ell \in \mathbb{Z}}$ of homomorphisms $h_\ell \colon X_\ell \to Y_{\ell+1}$, such that $\phi_\ell - \psi_\ell = \partial_{\ell+1}^Y h_\ell + h_{\ell-1}\partial_\ell^X$ for all $\ell \in \mathbb{Z}$. If such a homotopy exists, we say that ϕ and ψ are *homotopic*; this defines an equivalence relation of morphisms $X \to Y$. A morphism that is homotopic with the zero morphism is said to be *null-homotopic*.

The *mapping cone* of a morphism $\phi \colon X \to Y$ is the complex $\mathcal{M}(\phi)$ given by $\mathcal{M}(\phi)_\ell = Y_\ell \oplus X_{\ell-1} = (Y \oplus \Sigma X)_\ell$ and

$$\partial_\ell^{\mathcal{M}(\phi)} = \begin{pmatrix} \partial_\ell^Y & \phi_{\ell-1} \\ 0 & -\partial_{\ell-1}^X \end{pmatrix} : \begin{matrix} Y_\ell \\ \oplus \\ X_{\ell-1} \end{matrix} \longrightarrow \begin{matrix} Y_{\ell-1} \\ \oplus \\ X_{\ell-2} \end{matrix}$$

for all $\ell \in \mathbb{Z}$. An important property of the mapping cone is given in the following theorem (cf. [Fox98, (1.24)]).

Theorem 0.2. *If $\phi\colon X \to Y$ is a morphism of complexes, the (degreewise) inclusion $Y \hookrightarrow \mathcal{M}(\phi)$ and the (degreewise) projection $\mathcal{M}(\phi) \twoheadrightarrow \Sigma X$ are both morphisms of complexes, and together they form a short exact sequence*

$$0 \to Y \to \mathcal{M}(\phi) \to \Sigma X \to 0. \qquad \square$$

The mapping cone construction has other nice properties, including that of being natural and exact in the sense described by the easily established theorem below.

Theorem 0.3. *If there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & \tilde{X} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \tilde{\phi}} \\ Y & \xrightarrow{\lambda} & \tilde{Y} \end{array}$$

of complexes, then there is an induced morphism $\mathcal{M}(\phi) \to \mathcal{M}(\tilde{\phi})$ given in degree $\ell \in \mathbb{Z}$ by

$$\begin{pmatrix} \lambda_\ell & 0 \\ 0 & \gamma_{\ell-1} \end{pmatrix} : \mathcal{M}(\phi)_\ell = \begin{array}{c} Y_\ell \\ \oplus \\ X_{\ell-1} \end{array} \to \begin{array}{c} \tilde{Y}_\ell \\ \oplus \\ \tilde{X}_{\ell-1} \end{array} = \mathcal{M}(\tilde{\phi})_\ell.$$

If there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{X} & \xrightarrow{\bar{\gamma}} & X & \xrightarrow{\gamma} & \tilde{X} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{\phi}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \tilde{\phi}} & & \\ 0 & \longrightarrow & \bar{Y} & \xrightarrow{\bar{\lambda}} & Y & \xrightarrow{\lambda} & \tilde{Y} & \longrightarrow & 0 \end{array}$$

with exact rows, then the induced sequence $0 \to \mathcal{M}(\bar{\phi}) \to \mathcal{M}(\phi) \to \mathcal{M}(\tilde{\phi}) \to 0$ is exact. $\square$

All complexes together with all morphisms form a category denoted by $\mathcal{C}^R$. The subcategory of modules is denoted $\mathcal{C}_0^R$ for reasons that become obvious later. An important functor in the category $\mathcal{C}^R$ is the *homology functor* $\mathrm{H}\colon \mathcal{C}^R \to \mathcal{C}^R$, which takes a complex X to the complex $\mathrm{H}(X)$ defined by $\mathrm{H}(X)_\ell = \mathrm{H}_\ell(X) = \ker \partial_\ell^X / \operatorname{im} \partial_{\ell+1}^X$ and $\partial_\ell^{\mathrm{H}(X)} = 0$ for all $\ell \in \mathbb{Z}$, and a morphism $\phi\colon X \to Y$ to the morphism $\mathrm{H}(\phi)\colon \mathrm{H}(X) \to \mathrm{H}(Y)$ given by

$$\mathrm{H}(\phi)_\ell([x]_{\operatorname{im} \partial_{\ell+1}^X}) = \mathrm{H}_\ell(\phi)([x]_{\operatorname{im} \partial_{\ell+1}^X}) = [\phi_\ell(x)]_{\operatorname{im} \partial_{\ell+1}^Y}$$

for all $\ell \in \mathbb{Z}$ and $x \in \ker \partial_\ell^X$. $\mathrm{H}(X)$ is called the *homology complex* of X, and $\mathrm{H}_\ell(X)$ is called the *homology module in degree* ℓ. The complex X is *exact* if $\mathrm{H}(X) = 0$.

A short exact sequence $0 \to X \to Y \to Z \to 0$ of complexes induces an *exact sequence of homology modules* (cf. [Fox98, (1.21)]):

$$\cdots \to \mathrm{H}_{\ell+1}(Z) \to \mathrm{H}_\ell(X) \to \mathrm{H}_\ell(Y) \to \mathrm{H}_\ell(Z) \to \cdots .$$

If $\phi\colon X \to Y$ is a morphism of complexes, we say that ϕ is a *homology isomorphism*, and we write $\phi\colon X \xrightarrow{\simeq} Y$, if $\mathrm{H}(\phi)\colon \mathrm{H}(X) \to \mathrm{H}(Y)$ is an isomorphism; in other words,

$$X \xrightarrow[\phi]{\simeq} Y \overset{\text{def}}{\iff} \mathrm{H}(X) \xrightarrow[\mathrm{H}(\phi)]{\cong} \mathrm{H}(Y).$$

The following theorem presents a necessary and sufficient condition for a morphism to be a homology isomorphism (cf. [Fox98, Lemma 1.25]).

Theorem 0.4. *If $\phi\colon X \to Y$ is a morphism of complexes, then ϕ is a homology isomorphism if and only if its mapping cone $\mathcal{M}(\phi)$ is exact.* □

If R' is another (nontrivial, unitary and commutative) ring and $T\colon \mathcal{C}_0^R \to \mathcal{C}_0^{R'}$ is an additive, covariant functor, then T induces a functor $\mathcal{C}^R \to \mathcal{C}^{R'}$; the induced functor is also denoted by T. For any complex $X \in \mathcal{C}^R$, the complex $T(X)$ is given by $T(X)_\ell = T(X_\ell)$ and $\partial_\ell^{T(X)} = T(\partial_\ell^X)$ for all $\ell \in \mathbb{Z}$, and for any complex morphism $\phi\colon X \to Y$, the induced morphism $T(\phi)\colon T(X) \to T(Y)$ is given by $T(\phi)_\ell = T(\phi_\ell)$ for all $\ell \in \mathbb{Z}$. If, in addition, T is exact, that is, preserves short exact sequences of modules, then the induced functor commutes with homology, that is, $T(\mathrm{H}(X)) \cong \mathrm{H}(T(X))$ whenever X is a complex, and T preserves homology isomorphisms (cf. [Fox98, (1.51)]). In this case, in particular, T preserves the exactness of complexes.

Similarly, when $T\colon \mathcal{C}_0^R \to \mathcal{C}_0^{R'}$ is an additive, contravariant functor, there is an induced functor $\mathcal{C}^R \to \mathcal{C}^{R'}$ also denoted by T. For any complex X, the complex $T(X)$ is given by $T(X)_\ell = T(X_{-\ell})$ and $\partial_\ell^{T(X)} = -T(\partial_{-\ell+1}^X)$, and for any complex morphism $\phi\colon X \to Y$, the induced morphism $T(\phi)\colon T(Y) \to T(X)$ is given by $T(\phi)_\ell = T(\phi_{-\ell})$. Once again, if, in addition, T is an exact functor, then the induced functor commutes with homology and preserves homology isomorphisms.

The *tensor product* of two complexes X and Y is the complex $X \otimes_R Y$, whose ℓ'th module is

$$(X \otimes_R Y)_\ell = \coprod_{n \in \mathbb{Z}} X_n \otimes_R Y_{\ell-n},$$

and whose ℓ'th differential $\partial_\ell^{X \otimes_R Y}\colon (X \otimes_R Y)_\ell \to (X \otimes_R Y)_{\ell-1}$ is defined on generators $x_n \otimes y_{\ell-n} \in X_n \otimes_R Y_{\ell-n} \subseteq (X \otimes_R Y)_\ell$ by

$$\partial_\ell^{X \otimes_R Y}(x_n \otimes y_{\ell-n}) = \partial_n^X(x_n) \otimes y_{\ell-n} + (-1)^n x_n \otimes \partial_{\ell-n}^Y(y_{\ell-n}),$$

which is an element of $(X_{n-1} \otimes_R Y_{\ell-n}) \oplus (X_n \otimes_R Y_{\ell-n-1}) \subseteq (X \otimes_R Y)_{\ell-1}$. If X is concentrated in degree 0, then $X \otimes_R Y$ is the same as the complex obtained by applying the additive, covariant functor $X \otimes_R -$ to the complex Y as described in the preceding paragraph. Likewise, if Y is concentrated in degree 0, then $X \otimes_R Y$ is the same as the complex obtained by applying the additive, covariant functor $- \otimes_R Y$ to the complex X.

An important property of the tensor product of complexes is that if X and Y are complexes and S is a multiplicative system, then $S^{-1}(X \otimes_R Y)$ is isomorphic to $S^{-1}X \otimes_{S^{-1}R} S^{-1}Y$ as $S^{-1}R$-complexes (cf. [Fox98, (4.24)]).

Any property satisfied by the homology complex $\mathrm{H}(X)$ of a complex X is said to be satisfied *homologically* by X. If S is a multiplicative system in R, we say that X is S-torsion if $S^{-1}X = 0$; in particular, since $S^{-1}(-)$ is a covariant, exact functor, X is homologically S-torsion if and only if $S^{-1}X$ is exact. If $S = (S_1, \ldots, S_d)$ is a family of multiplicative systems, we say, as with modules, that X is S-torsion if X is S_ν-torsion for $\nu = 1, \ldots, d$.

Module invariants

One of the fundamental concepts within homological algebra is that of projective resolutions. A *resolution* of a module M is a complex

$$X = \quad \cdots \longrightarrow X_2 \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \longrightarrow 0,$$

together with a homomorphism $\phi_0 \colon X_0 \to M$, such that

$$\cdots \longrightarrow X_2 \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

is an exact complex; in other words, a resolution of M is a homology isomorphism $\phi \colon X \xrightarrow{\simeq} M$. If the modules in X are finitely generated, X is called a *finite resolution* of M; if the modules in X are projective, X is called a *projective resolution* of M; if the modules in X are flat, X is called a *flat resolution* of M; and if the modules in X are free, X is called a *free resolution* of M. We shall often combine the word “finite” with the words “projective” and “free”, defining a *finite projective resolution* to be a resolution that is finite as well as projective and a *finite free resolution* to be a resolution that is finite as well as free.

Any module has a free (and hence projective and flat) resolution, and if R is Noetherian and M is finitely generated, M has a finite free resolution (cf. [Fox98, Theorem 2.6]). If R is local with maximal ideal $\mathfrak{m}$ and M is finitely generated, we call X a *minimal free resolution* of M if it is a finite free resolution such that $\operatorname{im} \partial_\ell^X \subseteq \mathfrak{m}X_{\ell-1}$ for all $\ell \in \mathbb{Z}$.

If M is nontrivial, the *projective dimension* of M, denoted by $\mathrm{pd}_R\, M$, is the infimum over all $n \in \mathbb{N}_0$ for which M has a projective resolution X concentrated in degrees $n, \ldots, 0$; in particular, $\operatorname{pd}_R M = \infty$ if there are no bounded projective resolutions of M. Similarly the *flat dimension* of M, denoted by $\operatorname{fd}_R M$, is the infimum over all $n \in \mathbb{N}_0$ for which M has a flat resolution concentrated in degrees $n, \ldots, 0$. For technical reasons, we let $\operatorname{pd}_R 0 = \operatorname{fd}_R 0 = -\infty$, although we still consider the zero module to have finite projective and flat dimensions.

If M and N are modules and X is a projective resolution of M, application of the (covariant and right exact) functor $- \otimes_R N$ to X yields the complex

$$\cdots \longrightarrow X_2 \otimes_R N \xrightarrow{\partial_2^X \otimes_R N} X_1 \otimes_R N \xrightarrow{\partial_1^X \otimes N} X_0 \otimes_R N \longrightarrow 0$$

concentrated in nonnegative degrees. The homology module $\mathrm{H}_\ell(X \otimes_R N)$ of this is called the *ℓ'th Tor-module of M and N* and is denoted by $\operatorname{Tor}_\ell^R(M, N)$. Had we instead applied the (contravariant and left exact) functor $\operatorname{Hom}_R^\ell(-, N)$ to X, we would obtain the complex

$$0 \longrightarrow \operatorname{Hom}_R(X_0, N) \xrightarrow{\circ(-\partial_1^X)} \operatorname{Hom}_R(X_1, N) \xrightarrow{\circ(-\partial_2^X)} \operatorname{Hom}_R(X_2, N) \longrightarrow \cdots$$

concentrated in nonpositive degrees. The homology module $\mathrm{H}_{-\ell}(\operatorname{Hom}_R(X, N))$ of this is called the *ℓ'th Ext-module* and is denoted by $\operatorname{Ext}_R^\ell(M, N)$.

There is an alternative definition of $\operatorname{Tor}_\ell^R(M, N)$ using a projective resolution of N and an alternative definition of $\operatorname{Ext}_R^\ell(M, N)$ using an injective resolution of N, giving two possible interpretations of $\operatorname{Tor}_\ell^R(M, N)$ and $\operatorname{Ext}_R^\ell(M, N)$, but no ambiguities since the two interpretations are isomorphic. For our purposes, one definition suffices, so we will not give further mention to this in the sequel.

The zeroth Tor- and Ext-modules reflect the functors from which they were derived: if M and N are modules, $\operatorname{Tor}_0^R(M, N) \cong M \otimes_R N$ and $\operatorname{Ext}_R^0(M, N) \cong \operatorname{Hom}_R(M, N)$ (cf. [HS97, page 139 and 161]). Tor also maintains the property that the tensor product is commutative up to isomorphism, in the sense that $\operatorname{Tor}_\ell^R(M, N) \cong \operatorname{Tor}_\ell^R(N, M)$ for all $\ell \in \mathbb{Z}$. We also have that $\operatorname{Tor}_\ell^R(M, N)_\mathfrak{p} \cong \operatorname{Tor}_\ell^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ for any prime ideal $\mathfrak{p}$ (cf. [Eis95, page 161-162]); in particular,

$$\operatorname{Supp}_R(\operatorname{Tor}_\ell^R(M, N)) \subseteq \operatorname{Supp}_R M \cap \operatorname{Supp}_R N.$$

$\operatorname{Tor}_\ell^R(-, N)$, $\operatorname{Tor}_\ell^R(M, -)$ and $\operatorname{Ext}_R(M, -)$ are covariant functors, whereas $\operatorname{Ext}_R^\ell(-, N)$ is a contravariant functor. These functors take short exact sequences of modules to long exact sequences of modules: if A, B, C, M and N are modules and the sequence $0 \to A \to B \to C \to 0$ is exact, then the following sequences are exact and referred to as the *long exact* Tor- *and* Ext-*sequences* in the first and second variable (cf. [HS97, (7.1), (7.4), (11.1) and (11.4)]).

$$\cdots \to \operatorname{Tor}_{\ell+1}^R(M, C) \to \operatorname{Tor}_\ell^R(M, A) \to \operatorname{Tor}_\ell^R(M, B) \to \operatorname{Tor}_\ell^R(M, C) \to \cdots,$$
$$\cdots \to \operatorname{Tor}_{\ell+1}^R(C, N) \to \operatorname{Tor}_\ell^R(A, N) \to \operatorname{Tor}_\ell^R(B, N) \to \operatorname{Tor}_\ell^R(C, N) \to \cdots,$$
$$\cdots \to \operatorname{Ext}_R^{\ell-1}(M, C) \to \operatorname{Ext}_R^\ell(M, A) \to \operatorname{Ext}_R^\ell(M, B) \to \operatorname{Ext}_R^\ell(M, C) \to \cdots,$$
$$\cdots \to \operatorname{Ext}_R^{\ell-1}(A, N) \to \operatorname{Ext}_R^\ell(C, N) \to \operatorname{Ext}_R^\ell(B, N) \to \operatorname{Ext}_R^\ell(A, N) \to \cdots.$$

Projective dimension is naturally interlinked with Ext as described by the following theorem (cf. [Fox99, 9.1(1) and 9.3(1)]).

Theorem 0.5. *For a module M and an nonnegative integer n, the following conditions are equivalent.*

(i) $\operatorname{pd}_R M \leq n$.

(ii) $\operatorname{Ext}_R^m(M, -) = 0$ *for all* $m > n$.

(iii) $\operatorname{Ext}_R^{n+1}(M, -) = 0$.

(iv) *If there is an exact sequence* $0 \to X_n \to X_{n-1} \to \cdots X_0 \to M \to 0$ *in which* $X_0, \ldots, X_{n-1}$ *are projective, then* X_n *must be projective.*

In particular, $\operatorname{pd}_R M = \sup\{n \in \mathbb{N}_0 \,|\, \operatorname{Ext}_R^n(M, -) \neq 0\}$. □

Likewise, flat dimension is naturally interlinked with Tor as described by the following theorem (cf. [Fox99, 11.4(4) and 11.5(1)]).

Theorem 0.6. *For a module M and a nonnegative integer n, the following conditions are equivalent.*

(i) $\operatorname{fd}_R M \leq n$.

(ii) $\operatorname{Tor}_m^R(M, -) = 0$ *for all* $m > n$.

(iii) $\operatorname{Tor}_{n+1}^R(M, -) = 0$.

(iv) $\operatorname{Tor}_{n+1}^R(M, R/I) = 0$ *for all finitely generated ideals* I *in* R.

(v) *If there is an exact sequence* $0 \to X_n \to X_{n-1} \to \cdots X_0 \to M \to 0$ *in which* $X_0, \ldots, X_{n-1}$ *are flat, then* X_n *must be flat.*

In particular, $\operatorname{fd}_R M = \sup\{n \in \mathbb{N}_0 \,|\, \operatorname{Tor}_n^R(M, -) \neq 0\}$. □

In case R is Noetherian, flat and projective are the same for a finitely generated module M, and hence in this case $\operatorname{fd}_R M = \operatorname{pd}_R M$.

The following theorem allows us in some situations to determine the projective dimension of one of the modules in a short exact sequence from the projective dimension of the other two. (cf. [Fox99, HA071197-1(1)]):

Theorem 0.7. *Suppose* $0 \to L \to M \to N \to 0$ *is a short exact sequence of modules. Then if two of the modules have finite projective dimension, so does the third. In any case,* $\operatorname{pd}_R L = \operatorname{pd}_R N - 1$ *whenever* $\operatorname{pd}_R M < \operatorname{pd}_R N$. □

The module M is said to be *simple* if $M \neq 0$ and M has no nontrivial proper submodules: that is, if 0 is the only proper submodule of M. A *filtration* of M is a descending chain of submodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$$

of finite length n. If M is finitely generated, one can always find a filtration such that the factors M_{i-1}/M_i are isomorphic to $R/\mathfrak{p}$ for prime ideals $\mathfrak{p}$; the set of prime ideals thus obtained is contained in $\operatorname{Supp}_R M$ and contains $\operatorname{Ass}_R M$ (cf. [Rob98, Theorem 1.1.4]).

The *length* of a module M, denoted by $\operatorname{length}_R M$, is the supremum of the lengths of all filtrations $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ in which the inclusions are strict. One can show that, when the length of M is finite, the number $\operatorname{length}_R M$ is the length of any filtration of M in which the quotient modules M_{i-1}/M_i, $i = 1, \ldots, n$, are simple (cf. [Eis95, Theorem 2.13]); in this case, the filtration is called a *composition series* for M. If $0 \to L \to M \to N \to 0$ is a short exact sequence of modules, then $\operatorname{length}_R M = \operatorname{length}_R L + \operatorname{length}_R N$. (Here we follow the usual conventions that $n + \infty = \infty + n = \infty$ for all $n \in \mathbb{N}_0 \cup \{\infty\}$.) In addition, a module of finite length must be finitely generated. Note that if k is a field and V is a finitely generated k-vector space, $\operatorname{length}_k V = \operatorname{length}_R V$ is nothing but the usual dimension of V as a vector space over k. The notation "$\dim_k V$", however, is reserved for an important analog of dimension for modules over rings, which is described below.

Suppose that R is Noetherian and local with maximal ideal $\mathfrak{m}$ and quotient field $k = R/\mathfrak{m}$, and that M is a finitely generated module. The *n'th Betti number* of M is then the nonnegative integer $\beta_n^R(M) = \operatorname{length}_R \operatorname{Tor}_n^R(M, k)$: that is, the dimension of $\operatorname{Tor}_n^R(M, k)$ as a vector space over k. We know from Theorem 0.6 that $\operatorname{fd}_R M = \operatorname{pd}_R M = \sup\{n \in \mathbb{N}_0 \mid \beta_n^R(M) \neq 0\}$, and it follows that, if M has finite projective dimension, then $\beta_n^R(M) \neq 0$ for only finitely many n. In this case, the *Euler characteristic* of M is defined to be the alternating sum $\chi^R(M) = \sum_{n\in\mathbb{Z}}(-1)^n \beta_n^R(M)$. One can show that the n'th Betti number is the rank of the n'th module in a minimal free resolution of M and that the Euler characteristic equals the alternating sum $\sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{rank}_R X_\ell$ for *any* bounded finite free resolution X of M (cf. [Hal03, Proposition 4.2 and Theorem 5.2]).

The Euler characteristic has many nice properties. One is that it is always nonnegative, and another is that, if $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated modules of finite projective dimension, then $\chi^R(M) = \chi^R(L) + \chi^R(N)$ (cf. [Hal03, Proposition 5.6 and Theorem 5.5]). Theorem 0.8 below describes another interesting property of the Euler characteristic (cf. [Hal03, Theorem 5.7]).

Theorem 0.8. *If R is Noetherian and local, and M is a finitely generated module with $\operatorname{pd}_R M < \infty$, then the following conditions are equivalent.*

(i) $\chi^R(M) > 0$.

(ii) $\operatorname{Supp}_R M = \operatorname{Spec} R$.

(iii) $\operatorname{Ann}_R M = 0$.

(iv) $\operatorname{Ann}_R M \subseteq \operatorname{Zd} R$. $\square$

The definition of Euler characteristics can be generalized to complexes. If R is Noetherian and local and X is a complex such that there is a homology isomorphism $X \xleftarrow{\simeq} F \in \mathcal{C}^R_\square(\mathrm{f,P})$ (in which case we say that X has finite *projective dimension* and that F is a *bounded finite free resolution* of X), then the *Euler characteristic* of X is the sum $\chi^R(X) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{length}_R \mathrm{H}_\ell(k \otimes_R F)$, which is equal to the sum $\sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{rank}_R F_\ell$ (cf. [Hal03, Theorem 5.2]). From this, it follows that, if $\mathfrak{p}$ is a prime ideal, then $\chi^R(X) = \chi^{R_\mathfrak{p}}(X_\mathfrak{p})$.

The *height* of a prime ideal $\mathfrak{p}$, denoted by $\operatorname{height}_R \mathfrak{p}$, is the supremum of nonnegative integers n for which there is a strictly descending chain

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$$

of prime ideals. For an arbitrary ideal I, one defines the height of I by

$$\operatorname{height}_R I = \inf\{\operatorname{height}_R \mathfrak{p} \,|\, \mathfrak{p} \in V_R(I)\}.$$

The *(Krull) dimension* of M, denoted by $\dim_R M$, is the supremum of the lengths of all strictly descending chains $\mathfrak{p}_0 \supset \mathfrak{p}_1 \cdots \supset \mathfrak{p}_n$ of prime ideals in $\operatorname{Supp}_R M$. If M is finitely generated, $\operatorname{Supp}_R M = V_R(\operatorname{Ann}_R M)$, so in this case we must have $\dim_R M = \dim_R R/\operatorname{Ann}_R M$. Considering R as a module over itself, we see that the Krull dimension of R is

$$\dim R = \sup\{\operatorname{height}_R \mathfrak{p} \,|\, \mathfrak{p} \in \operatorname{Spec} R\},$$

and it follows that $\dim R_\mathfrak{p} = \operatorname{height}_R \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. In particular, when R is local with maximal ideal $\mathfrak{m}$, $\dim R = \operatorname{height}_R \mathfrak{m}$.

A fundamental theorem about heights is *Krull's principal ideal theorem* presented below (cf. [BH93, Theorem A.1]).

Theorem 0.9 (Krull). *If R is Noetherian and $\mathfrak{p}$ is a minimal prime over an ideal I generated by n elements, then* $\operatorname{height}_R \mathfrak{p} \leq n$. $\square$

Krull's principal ideal theorem shows that the height of any ideal is bounded by the number of generators for that ideal. In particular, when R is Noetherian and local, the Krull dimension of R (and hence of any R-module) is finite.

Krull dimension and length are integrated in the following theorem (cf. [Eis95, Theorem 2.17 and Lemma 2.8b]).

Theorem 0.10. *If R is Noetherian and M is a finitely generated module, then the following are equivalent.*

(i) $\dim_R M = 0$.

(ii) $\operatorname{Supp}_R M$ *is nonempty and contains only maximal ideals.*

(iii) $0 < \operatorname{length}_R M < \infty$. □

The following theorem relates dimension to height (cf. [BH93, page 413]).

Theorem 0.11. *If I is a proper ideal, then* $\operatorname{height}_R I + \dim_R R/I \leq \dim R$. □

Suppose R is Noetherian and local with maximal ideal $\mathfrak{m}$, and let M be a nontrivial and finitely generated module. Then $\dim_R M$ is the minimum of all $d \in \mathbb{N}_0$ for which there exist elements $x_1, \ldots, x_d \in \mathfrak{m}$ such that $M/\langle x_1, \ldots, x_d \rangle M$ has finite length, and in this case, the sequence $(x_1, \ldots, x_d)$ is called a *system of parameters* for M (cf. [Rob98, Theorem 2.3.7]). Systems of parameters are characterized by the following theorem (cf. [BH93, Proposition A.4]).

Theorem 0.12. *If R is Noetherian and local with maximal ideal $\mathfrak{m}$, M is a nontrivial and finitely generated module, and $x_1, \ldots, x_n$ are elements in $\mathfrak{m}$, then*

$$\dim_R(M/\langle x_1, \ldots, x_n \rangle M) \geq \dim_R M - n$$

with equality holding if and only if $x_1, \ldots, x_n$ is part of a system of parameters for M. □

If R is Noetherian and local with maximal ideal $\mathfrak{m}$, R is said to be *regular* if it has a system of parameters generating $\mathfrak{m}$; such a system of parameters is called a *regular system of parameters.* A Noetherian local ring is regular if and only if every finitely generated module has finite projective dimension, and a regular local ring is always an integral domain (cf. [BH93, Theorem 2.2.7 and Proposition 2.2.3]). If R equals the quotient Q/I of a regular local ring Q with an ideal I of Q generated by $\operatorname{height}_Q I$ elements, then we say that R is a *complete intersection.*[1] This definition clearly shows that any regular local ring is a complete intersection.

If M is a module, an *M-regular sequence*, or simply an *M-sequence*, is a sequence $x = (x_1, \ldots, x_n)$ of elements of R such that $\langle x \rangle M \neq M$ and such that x_i is a non-zerodivisor on $M/\langle x_1, \ldots, x_{i-1} \rangle M$ for $i = 1, \ldots, n$. In the case $i = 1$, the last condition simply says that x_1 must be a non-zerodivisor on M. The sequence $x = (x_1, \ldots, x_n)$ is called a *weak M-sequence* if it only satisfies the latter condition: that x_i is a non-zerodivisor on $M/\langle x_1, \ldots, x_{i-1} \rangle M$ for $i = 1, \ldots, n$. An R-sequence is simply called a *regular sequence.*

[1] This is a more restricted definition than that traditionally given. Normally, one defines a complete intersection to be a ring whose completion is the quotient of a regular local ring by a regular sequence. This definition, however, suffices for the present purposes, and using it avoids having to introduce the concept of completion of rings.

If R is Noetherian, M is finitely generated and $x = (x_1 \dots, x_n)$ is an M-sequence contained in a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R M$, then $x/1 = (x_1/1, \dots, x_n/1)$ is an $M_\mathfrak{p}$-sequence contained in $\mathfrak{p}_\mathfrak{p}$ (cf. [BH93, Corollary 1.1.3(i)]). If, in addition, R is local, the next theorem shows that the elements of a regular sequence can be permuted and raised to powers (cf. [BH93, Proposition 1.1.6]).

Theorem 0.13. *If R is Noetherian and local, M is a finitely generated module, and $x = (x_1, \dots, x_n)$ is an M-sequence, then every permutation of x is an M-sequence, and so is $(x_1^{N_1}, \dots, x_n^{N_n})$ for any $N_1, \dots, N_n \in \mathbb{N}$.* □

A *maximal* M-sequence in an ideal I of R is an M-sequence $x = (x_1, \dots, x_n)$ contained in I, such that $(x_1, \dots, x_n, x_{n+1})$ is not an M-sequence for any $x_{n+1} \in I$. Under the conditions of the next theorem, all maximal M-sequences in I have the same length (cf. [BH93, Theorem 1.2.5]).

Theorem 0.14 (Rees). *If R is Noetherian, M is a finitely generated module and I is an ideal such that $IM \neq M$, then all maximal M-sequences in I have the same length n given by $n = \min\{i \in \mathbb{N}_0 \,|\, \operatorname{Ext}^i_R(R/I, M) \neq 0\}$.* □

Whenever the conditions of the theorem hold, one defines the *grade of I on M*, denoted by $\operatorname{grade}_R(I, M)$, to be the common length of all maximal M-sequences in I. In case $IM = M$, we complement the definition by setting $\operatorname{grade}_R(I, M) = \infty$; this is consistent with Theorem 0.14, since

$$\operatorname{grade}_R(I, M) = \infty \iff IM = M \iff \operatorname{Ext}^i_R(R/I, M) = 0 \text{ for all } i \in \mathbb{N}_0$$

(cf. [BH93, page 10]). Thus, we have

$$\operatorname{grade}_R(I, M) = \inf\{i \in \mathbb{N}_0 \,|\, \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

With this in mind, one defines the *grade of M* by

$$\operatorname{grade}_R M = \inf\{i \in \mathbb{N}_0 \,|\, \operatorname{Ext}^i_R(M, R) \neq 0\}.$$

From Theorem 0.5 we immediately derive that $\operatorname{grade}_R M \leq \operatorname{pd}_R M$ whenever R is Noetherian and M is nontrivial. For a nonnegative integer p, we say that M is *p-perfect* if $\operatorname{grade}_R M = p = \operatorname{pd}_R M$ or $M = 0$.

If R is Noetherian and local with maximal ideal $\mathfrak{m}$ and M is a finitely generated module, we refer to $\operatorname{grade}_R(\mathfrak{m}, M)$ as the *depth* of M and denote it by $\operatorname{depth}_R M$. Depth and dimension come together in the following proposition (cf. [BH93, Proposition 1.2.12]).

Proposition 0.15. *If R is Noetherian and local, and M is a nontrivial, finitely generated module, then every M-sequence is part of a system of parameters for M. In particular, $\operatorname{depth}_R M \leq \dim_R M$.* □

If R is Noetherian and local, and M is a nontrivial, finitely generated module, we say that M is *Cohen–Macaulay* if $\operatorname{depth}_R M = \dim_R M$; in particular, R is Cohen–Macaulay if $\operatorname{depth} R = \dim R$. Any complete intersection ring is Cohen–Macaulay (cf. [BH93, Proposition 3.1.20]).

Depth is also interlinked with projective dimension in the famous *Auslander–Buchsbaum formula* (cf. [BH93, Theorem 1.3.3]):

Theorem 0.16 (Auslander–Buchsbaum). *If R is Noetherian and local, and M is a nontrivial, finitely generated module with $\operatorname{pd}_R M < \infty$, then*

$$\operatorname{pd}_R M + \operatorname{depth}_R M = \operatorname{depth} R. \qquad \square$$

Another interesting theorem related to depth is the following lemma by Peskine and Szpiro, which is commonly known as the *acyclicity lemma* (cf. [Eis95, Lemma 20.11]).

Lemma 0.17 (acyclicity). *Suppose that R is Noetherian and local, and that $X = 0 \to X_n \to \cdots \to X_1 \to X_0 \to 0$ is a complex of finitely generated modules such that $\operatorname{depth}_R X_\ell \geq \ell$ for $\ell = 0, \ldots, n$. Then, if $\mathrm{H}_\ell(X) \neq 0$ for some $\ell > 0$, then for the largest such ℓ we have $\operatorname{depth}_R \mathrm{H}_\ell(X) \geq 1$.*

Although grade is not additive on short exact sequences, a lower bound for the grade of one of the modules in a short exact sequence can always be estimated if the grades of the two others are known (cf. [BH93, Proposition 1.2.9]):

Proposition 0.18. *If R is Noetherian, I is an ideal, and $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated modules, then*

(i) $\operatorname{grade}_R(I, A) \geq \min\{\operatorname{grade}_R(I, B), \operatorname{grade}_R(I, C) + 1\}$.

(ii) $\operatorname{grade}_R(I, B) \geq \min\{\operatorname{grade}_R(I, A), \operatorname{grade}_R(I, C)\}$.

(iii) $\operatorname{grade}_R(I, C) \geq \min\{\operatorname{grade}_R(I, A) - 1, \operatorname{grade}_R(I, B)\}$. $\square$

The proposition below compiles other nice properties of grade (cf. [BH93, Proposition 1.2.10]).

Proposition 0.19. *If R is Noetherian, I and J are ideals and M is a finitely generated module, then*

(i) $\operatorname{grade}_R(I, M) = \inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \,|\, \mathfrak{p} \in V_R(I)\}$.

(ii) $\operatorname{grade}_R(I, M) = \operatorname{grade}_R(\operatorname{Rad} I, M)$.

(iii) $\operatorname{grade}_R(I \cap J, M) = \min\{\operatorname{grade}_R(I, M), \operatorname{grade}_R(J, M)\}$.

(iv) *If $x = (x_1, \ldots, x_n)$ is an M-sequence in I, then*

$$\operatorname{grade}_R(I/\langle x\rangle, M/\langle x\rangle M) = \operatorname{grade}_R(I, M/\langle x\rangle M) = \operatorname{grade}_R(I, M) - n.$$

(v) *If N is a finitely generated module with* $\operatorname{Supp}_R N = V_R(I)$, *then*

$$\operatorname{grade}_R(I, M) = \inf\{i \,|\, \operatorname{Ext}^i_R(N, M) \neq 0\}. \qquad \square$$

If R is Noetherian and M is finitely generated, it follows immediately from part (v) of Proposition 0.19 that $\operatorname{grade}_R M = \operatorname{grade}_R(\operatorname{Ann}_R M, R)$. If, in addition, R is local and M is nontrivial, part (i) together with Proposition 0.15 imply that $\operatorname{grade}_R M \leq \operatorname{height} \operatorname{Ann}_R M$, and it then follows from Theorem 0.11 that $\operatorname{grade}_R M + \dim_R M \leq \dim R$. The *grade conjecture* states that if, in addition, $\operatorname{pd}_R M < \infty$, this is actually an equality. Theorem 0.8 verifies the grade conjecture in the case that $\operatorname{grade}_R M = 0$, since we apparently have

$$\operatorname{grade}_R M = 0 \iff \chi^R(M) \neq 0 \iff \dim_R M = \dim R.$$

We shall later verify the grade conjecture in the case that $\operatorname{grade}_R M = 1$.

Chapter 1

Introduction

One of the fundamental results within algebraic geometry is that of Bézout's theorem, stating that if two plane algebraic curves of degree m and n, respectively, in complex projective plane have no common components, then the number of intersection points of the two curves when counted with multiplicity is mn. Here the "multiplicity" of an intersection point of two curves is obtained by calculating the dimension of a certain complex vector space associated with the polynomials defining the curves. Substantial work has been put into defining intersection multiplicities in a more general and strictly algebraic setting so that Bézout's theorem is satisfied. Serre introduced such a generalization in [Ser65] more than 40 years ago.

Let R be a regular local ring, and let M and N be finitely generated R-modules such that $M \otimes_R N$ has finite length. Under these hypotheses, the module $\operatorname{Tor}_\ell^R(M,N)$ has finite length for all ℓ and is zero for all but finitely many ℓ, and one can therefore define the *intersection multiplicity* of M and N to be the number

$$\chi^R(M,N) = \sum_{\ell=0}^{\infty} (-1)^\ell \operatorname{length}_R \operatorname{Tor}_\ell^R(M,N).$$

Having presented this new definition, Serre proved that it satisfied many of the properties required by intersection multiplicities, including Bézout's theorem. Serre also conjectured that the following conditions hold.

(0) $\dim_R M + \dim_R N \leq \dim R$,

(1) $\chi^R(M,N) \geq 0$,

(2) $\chi^R(M,N) \neq 0$ if and only if $\dim_R M + \dim_R N = \dim R$.

These have later been known as the *intersection conjectures*. They are part of a collection of conjectures known as the *homological conjectures*, originally introduced by Hochster, among which we find the *grade conjecture*, which is also discussed briefly in this thesis.

Serre proved that condition (0) holds in general and that all the conjectures hold in many cases (namely for equicharacteristic rings and unramified rings of mixed characteristics—two concepts that are not treated in this thesis). The conjectures and generalizations of them have been among the central problems in commutative algebra since they were published.

The second and third condition can be replaced by

(1′) $\chi^R(M, N) = 0$ if $\dim_R M + \dim_R N < \dim R$,

(2′) $\chi^R(M, N) > 0$ if $\dim_R M + \dim_R N = \dim R$.

The claim that condition (1) of the original statement holds is known as the *nonnegativity conjecture*. The claim that condition (1′) above holds is known as the *vanishing conjecture*, and the claim that condition (2′) holds is known as the *positivity conjecture*.

Great progress has been achieved since Serre proposed his conjectures. The vanishing conjecture was proven in about 1985 by Roberts in [Rob85] and, independently, by Gillet and Soulé, and Gabber proved the nonnegativity conjecture in about 1996. So of the original intersection conjectures, only positivity remains open—in fact, it only remains to be shown that $\chi^R(M, N) \neq 0$ whenever $\dim_R M + \dim_R N = \dim R$ (and only in the case where R is ramified and of mixed characteristics). However, many other questions remain unanswered as to whether the intersection conjectures hold in more general settings where it makes sense to define the intersection multiplicity of modules M and N over a ring R.

One such typical generalization is to replace the assumption that R is regular by the weaker assumption that R is Noetherian and local and that M and N have finite projective dimensions. Under these assumptions, the intersection conjectures are still open. One need only assume that one of the modules has finite projective dimension to define the intersection multiplicity, but as shown by a counterexample discovered in 1985 by Dutta, Hochster and McLaughlin, vanishing and positivity fail in this generality. However, if one of the modules has finite projective dimension and the other has (Krull) dimension 0 or 1, the intersection conjectures hold; Foxby showed this in [Fox82b] in about 1981.

The purpose of this thesis is to present the algebraic K-theory behind Foxby's proof of the intersection conjectures; to outline the methods used by Roberts in his proof of the vanishing conjecture; and to discuss how these two apparently unrelated approaches are interlinked. The intersection multiplicity is naturally investigated in a K-theoretical setting, since $\chi^R(M, -)$ and $\chi^R(-, N)$ are additive on short exact sequences of modules for which they are defined and hence factor through the Grothendieck groups of categories of such modules. In his proof, Foxby exploited an isomorphism between certain Grothendieck groups to show that a calculation of $\chi^R(M, N)$ can be reduced to the case where one replaces M by $R/G_R(M)$, where $G_R(M)$ is a principal ideal originally introduced by MacRae and referred to in his honor as the *MacRae ideal*. Since some of the properties of this ideal are known, the intersection conjectures are then seen to hold.

Roberts took another approach in his proof of the vanishing conjecture, using local Chern characters. Although the mere definition of local Chern characters is quite unaccessible and lies beyond the scope of this thesis, they satisfy a number of very nice functorial conditions, allowing one to present the intersection multiplicity of two modules as an element in a $\mathbb{Q}$-vector space obtained by an application of the local Chern characters associated to projective resolutions of the two modules. When the ring is a complete intersection (which is the case if it is a regular local ring), simplification is possible and the vanishing conjecture is seen to hold.

The relationship between the two approaches to the intersection conjectures is apparent from the role played by the Euler characteristic and the MacRae ideal. Not only do these two invariants appear in the simplifications of modules in the Grothendieck groups Foxby used in his proof, they also uniquely determine the zeroth and first local Chern characters of projective resolutions of modules. This fact gives hope that, perhaps in future research, the discovery of a new invariant, related to the second local Chern character, will allow the intersection conjectures to be proven in one more case.

Chapter 2

Groups of complexes: K_0

Vector spaces over a field are classified according to their dimension. This classification is sufficiently coarse-grained that each vector space is represented by something as simple as a cardinal number and sufficiently fine-grained that each vector space is uniquely determined up to isomorphism by its dimension.

The "weaker" structure of a ring compared with that of a field makes constructing a similarly powerful classification of modules over a ring impossible. A multitude of coarse-grained classifications of modules do exist, however, using invariants such as length, depth, grade, Krull dimension, projective dimension and Euler characteristic. At the other end of the scale we find the ultimate fine-grained classification of modules, simply taking each module to its isomorphism class. Somewhere before that we encounter the Grothendieck groups.

If $\mathrm{c}(V)$ denotes the isomorphism class of a finitely generated vector space V over a field k, assigning to $\mathrm{c}(V)$ the vector space dimension of V yields a one-to-one correspondence between the collection $\mathcal{I}$ of isomorphism classes of finitely generated k-vector spaces and the set $\mathbb{N}_0$ of nonnegative integers. The addition on $\mathbb{N}_0$ is thus imposed on $\mathcal{I}$, making it an Abelian monoid with identity element $\mathrm{c}(0)$ and addition given by $\mathrm{c}(V)+\mathrm{c}(W)=\mathrm{c}(V\oplus W)$. By including all differences $\mathrm{c}(V)-\mathrm{c}(W)$, $\mathcal{I}$ is completed to an Abelian group $K_0(k)$ isomorphic to $\mathbb{Z}$; this is the *Grothendieck group* of k.

In the following sections, this construction will be generalized to modules and complexes over rings, thereby yielding the Grothendieck groups.

2.1 Categories of complexes

This thesis covers multiple categories of complexes. The notation used has the advantage that it is very flexible and thus can be applied to describe a vast number of categories of complexes. Furthermore, the symbols involved are very descriptive, so that one can almost guess what category a certain symbol represents without having to look it up in the definition.

Definition 2.1. We define the following full subcategories of the category $\mathcal{C}^R$ of complexes by specifying their objects:

$\mathcal{C}^R_\square$: complexes that are bounded;

$\mathcal{C}^R_{[b,a]}$: complexes that are concentrated in degrees $b, b-1, \dots, a$; here b, a are integers with $b \geq a$;

$\mathcal{C}^R_{]\infty,a]}$: complexes that are concentrated in degrees $b, b-1, \dots, a$ for some b; here b, a are integers with $b \geq a$;

$\mathcal{C}^R_c$: complexes that are concentrated in degrees $c, c-1, \dots, 0$; here c is a nonnegative integer;

$\mathcal{C}^R(\mathrm{f})$: complexes of finitely generated modules;

$\mathcal{C}^R(\mathrm{l})$: complexes of modules of finite length;

$\mathcal{C}^R(\mathrm{P})$: complexes of projective modules;

$\mathcal{C}^R(\mathrm{F})$: complexes of free modules;

$\mathcal{C}^R(\mathrm{pd})$: complexes of modules of finite projective dimension;

$\mathcal{C}^R(S\text{-tor})$: complexes of S-torsion modules; here $S = (S_1, \dots, S_d)$ is a family of multiplicative systems;

$\mathcal{C}^R(\mathrm{gr} \geq g)$: complexes of modules of grade greater than or equal to g; here g is a nonnegative integer; and

$\mathcal{C}^R(p\text{-perf})$: complexes X of p-perfect modules; here p is a nonnegative integer.

(Such terms as "concentrated", "S-torsion", "grade" and "p-perfect" are defined in the preliminaries.)

This list could be continued indefinitely by introducing new symbols to describe the "shape" of the complexes and the properties of the modules in the complexes. Here we have only included the symbols that are used in the sequel.

We will freely use any combination of the symbols above by setting

$$\mathcal{C}^R_\star(\#_1, \dots) \stackrel{\text{def}}{=} \mathcal{C}^R_\star \cap \mathcal{C}^R(\#_1) \cap \dots,$$

so that, for example, $\mathcal{C}^R_\square(\mathrm{f,P})$ denotes the category of bounded complexes of finitely generated projective modules. (Note that we do not consider the case where we have several of the subscripts "$\star$", since any combination of subscripts can be expressed by a single one of them.)

The homology functor H entails a new series of subcategories of $\mathcal{C}^R$. These are defined by

$$X \in \mathcal{C}^R_\star(\#_1, \dots | \#'_1, \dots) \stackrel{\text{def}}{\iff} X \in \mathcal{C}^R_\star(\#_1, \dots) \text{ and } \mathrm{H}(X) \in \mathcal{C}^R(\#'_1, \dots),$$

so that, for instance, $\mathcal{C}^R_\square(\mathrm{f,P}|S\text{-tor})$ denotes the category of bounded, homologically S-torsion complexes of finitely generated projective modules.

Before introducing the Grothendieck groups, some technical problems concerning isomorphism classes of objects in a category need to be cleared up. The problem is that an isomorphism class need not be a set, nor does the collection of isomorphism classes (even if each isomorphism class, in fact, is a set). Both of these set-theoretical problems can be overcome if the category is chosen sufficiently "small" that its objects can be represented by a set.

Definition 2.2. Let $\mathcal{C}$ be a category and suppose $\mathcal{O}$ is a set of objects of $\mathcal{C}$. We say that $\mathcal{C}$ *is represented by* $\mathcal{O}$ if $\mathcal{O}$ contains an object from each isomorphism class of $\mathcal{C}$. A category $\mathcal{C}$ is said to be *modest* if it is represented by a set. In this case, the *restricted isomorphism class* of an object A of $\mathcal{C}$ is the nonempty set $\mathrm{c}(A) = \{B \in \mathcal{O} \,|\, A \cong B\}$ consisting of elements of $\mathcal{O}$ that are in the isomorphism class of A. The set of restricted isomorphism classes of $\mathcal{C}$ will be denoted by $\mathcal{I}(\mathcal{C})$.

Henceforth, we shall suppress the word "restricted" and simply refer to $\mathrm{c}(A)$ as the isomorphism class of A and to $\mathcal{I}(\mathcal{C})$ as the set of isomorphism classes of $\mathcal{C}$.

The categories considered here are all modest, since they are all full subcategories of the category of bounded complexes of finitely generated modules. We state this as a proposition, but leave the (easy) proof as an exercise for those interested in set theory.

Proposition 2.3. *The category* $\mathcal{C}^R_\square(\mathrm{f})$ *of bounded complexes of finitely generated modules is modest, and so is any full subcategory of* $\mathcal{C}^R_\square(\mathrm{f})$. □

Note the importance of the requirement in the above proposition that a subcategory $\mathcal{C}$ of $\mathcal{C}^R_\square(\mathrm{f})$ must be *full* to be modest. Constructing a subcategory of $\mathcal{C}^R_\square(\mathrm{f})$ that is not modest is easy: the subcategory consisting of all the objects of $\mathcal{C}^R_\square(\mathrm{f})$ and no morphisms but the identities is not modest, since no two distinct objects are isomorphic and the collection of objects do not form a set.

2.2 Grothendieck groups

Now that we have avoided getting ourselves into any kind of set-theoretical problems, we are finally ready to present the Grothendieck groups.

Definition 2.4. Suppose $\mathcal{C}$ is a full subcategory of $\mathcal{C}^R_\square(\mathrm{f})$. The *Grothendieck group* $K_0(\mathcal{C})$ of $\mathcal{C}$ is the group presented by generators $[X]$, one for each isomorphism class $\mathrm{c}(X)$, and relations $[X] = 0$ whenever X is exact and $[Y] = [X] + [Z]$ whenever there is an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$; in other words, $K_0(\mathcal{C}) = F_{\mathbb{Z}}(\mathcal{I}(\mathcal{C}))/\mathcal{E}(\mathcal{C})$, where $F_{\mathbb{Z}}(\mathcal{I}(\mathcal{C}))$ is the free Abelian group based on the set $\mathcal{I}(\mathcal{C})$ of isomorphism classes of $\mathcal{C}$, and $\mathcal{E}(\mathcal{C})$ is the subgroup generated by elements $\mathrm{c}(X)$ for which X is exact and elements $\mathrm{c}(Y) - \mathrm{c}(X) - \mathrm{c}(Z)$ for which there is a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$.

Two categories are important enough that their Grothendieck groups have special names and notation in the literature. The *Grothendieck group of* R is given by $G_0(R) \overset{\mathrm{def}}{=} K_0(\mathcal{C}^R_0(\mathrm{f}))$, and the *zeroth algebraic* K*-group of* R is given by $K_0(R) \overset{\mathrm{def}}{=} K_0(\mathcal{C}^R_0(\mathrm{f,P}))$. This thesis, however, investigates many other Grothendieck groups, and for these we shall follow the notation from Definition 2.1.

Definition 2.5. If $\mathcal{C}^R_\star(\#_1, \dots | \#'_1, \dots)$ is one of the categories from Definition 2.1, we denote its Grothendieck group by $G^R_\star(\#_1, \dots | \#'_1, \dots)$.

If $\mathcal{C}$ is a full subcategory of $\mathcal{C}^R_\square(\mathrm{f})$ and $\mathcal{C}$ does not contain the zero complex, then there are no exact sequences in $K_0(\mathcal{C})$, so in this case we have $K_0(\mathcal{C}) = F_\mathbb{Z}(\mathcal{I}(\mathcal{C}))/\mathcal{E}(\mathcal{C})$, where $\mathcal{E}(\mathcal{C})$ is the subgroup generated by isomorphism classes $\mathrm{c}(X)$ for which X is exact; in particular, if $\mathcal{C}$ is a subcategory of the category $\mathcal{C}^R_0(\mathrm{f})$ of finitely generated modules, then $K_0(\mathcal{C})$ is the free Abelian group $F_\mathbb{Z}(\mathcal{I}(\mathcal{C}))$. On the other hand, if the zero complex is an object of $\mathcal{C}$, the short exact sequence $0 \to 0 \to 0 \to 0 \to 0$ of complexes shows that $[0] = 0$ in $K_0(\mathcal{C})$. All the categories of complexes treated in this thesis contain the zero complex.

The fact that the existence of a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$ forces the equation $[Y] = [X] + [Z]$ to hold in $K_0(\mathcal{C})$ means, in particular, that $[X \oplus Z] = [X] + [Z]$ whenever $0 \to X \to X \oplus Z \to Z \to 0$ is an exact sequence in $\mathcal{C}$. Consequently, if $\mathcal{C}$ is a full subcategory of $\mathcal{C}^R_\square(\mathrm{f})$ that is closed under direct sum (which is the case for all the categories considered here), then any finite sum of "nonnegative" elements in $K_0(\mathcal{C})$ collapses to a single term, and it follows that any element of $K_0(\mathcal{C})$ can be written as $[X] - [X']$ for complexes $X, X' \in \mathcal{C}$. The following proposition enables us to determine when such an element is trivial.

Proposition 2.6. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}^R_\square(\mathrm{f})$ containing the zero complex, and suppose that $\mathcal{C}$ is closed under direct sum. Furthermore, let X and X' be complexes in $\mathcal{C}$. Then $[X] = [X']$ in $K_0(\mathcal{C})$ if and only if $\mathcal{C}$ contains exact complexes Y and Y' and short exact sequences $0 \to U \to V \to W \to 0$ and $0 \to U \to V' \to W \to 0$ such that $X \oplus Y' \oplus V' \cong X' \oplus Y \oplus V$.*

PROOF: "if" is clear, since in this case we have $[V'] = [U] + [W] = [V]$ and

$$\begin{aligned}
[X] + [V'] &= [X] + [Y'] + [V'] \\
&= [X \oplus Y' \oplus V'] \\
&= [X' \oplus Y \oplus V] \\
&= [X'] + [Y] + [V] \\
&= [X'] + [V].
\end{aligned}$$

For the other direction, suppose that X and X' are complexes in $\mathcal{C}$ such that $[X] = [X']$ in $K_0(\mathcal{C})$. Then the element $\mathrm{c}(X) - \mathrm{c}(X') \in F_\mathbb{Z}(\mathcal{I}(\mathcal{C}))$ is in the subgroup $\mathcal{E}(\mathcal{C})$ described in Definition 2.4, and hence we can write

$$\begin{aligned}
\mathrm{c}(X) - \mathrm{c}(X') = &\sum_i \mathrm{c}(Y_i) - \sum_j \mathrm{c}(Y'_j) \\
&+ \sum_k (\mathrm{c}(V_k) - \mathrm{c}(U_k) - \mathrm{c}(W_k)) - \sum_\ell (\mathrm{c}(V'_\ell) - \mathrm{c}(U'_\ell) - \mathrm{c}(W'_\ell)),
\end{aligned}$$

where Y_i and Y'_j are exact for all i and j, and there are short exact sequences $0 \to U_k \to V_k \to W_k \to 0$ and $0 \to U'_\ell \to V'_\ell \to W'_\ell \to 0$ for all k and ℓ, respectively. Moving negative terms to the other side of the equation, we obtain

$$\begin{aligned} &\mathrm{c}(X) + \sum_j \mathrm{c}(Y_j') + \sum_k (\mathrm{c}(U_k) + \mathrm{c}(W_k)) + \sum_\ell \mathrm{c}(V_\ell') \\ &\qquad = \mathrm{c}(X') + \sum_i \mathrm{c}(Y_i) + \sum_k \mathrm{c}(V_k) + \sum_\ell (\mathrm{c}(U_\ell') + \mathrm{c}(W_\ell')). \end{aligned}$$

This is an equation within the free Abelian group $F_{\mathbb{Z}}(\mathcal{I}(\mathcal{C}))$. The terms on each side of the equation are all elements of the generating set, so the fact that the equation holds implies that there is termwise equality. In particular, taking the direct sum of the terms on each side turns the above equation into an isomorphism

$$\begin{aligned} &X \oplus (\coprod_j Y_j') \oplus (\coprod_k (U_k \oplus W_k)) \oplus (\coprod_\ell V_\ell') \\ &\qquad \cong X' \oplus (\coprod_i Y_i) \oplus (\coprod_k V_k) \oplus (\coprod_\ell (U_\ell' \oplus W_\ell')). \end{aligned}$$

The direct sums $Y' = \coprod_j Y_j'$ and $Y = \coprod_i Y_i$ are exact since their summands are. Furthermore, the direct sum of short exact sequences of complexes is again a short exact sequence, so letting

$$\begin{aligned} \tilde{U} &= \coprod_k U_k, & \tilde{V} &= \coprod_k V_k, & \tilde{W} &= \coprod_k W_k, \\ \tilde{U}' &= \coprod_\ell U_\ell', & \tilde{V}' &= \coprod_\ell V_\ell', & \tilde{W}' &= \coprod_\ell W_\ell', \end{aligned}$$

we have short exact sequences

$$0 \to \tilde{U} \oplus \tilde{U}' \to \tilde{V} \oplus \tilde{U}' \oplus \tilde{W}' \to \tilde{W} \oplus \tilde{W}' \to 0$$

and

$$0 \to \tilde{U}' \oplus \tilde{U} \to \tilde{V}' \oplus \tilde{U} \oplus \tilde{W} \to \tilde{W}' \oplus \tilde{W} \to 0.$$

Since $\mathcal{C}$ is closed under direct sum, we can now let $U = \tilde{U} \oplus \tilde{U}'$, $W = \tilde{W} \oplus \tilde{W}'$, $V = \tilde{U} \oplus \tilde{W} \oplus \tilde{V}'$ and $V' = \tilde{V} \oplus \tilde{U}' \oplus \tilde{W}'$, and the proposition is proved. □

In the traditional zeroth algebraic K-group $K_0(R)$, things are much nicer.

Corollary 2.7. *If M and N are modules in $\mathcal{C}_0^R$(f,P), then $[M] = [N]$ in $K_0(R)$ if and only if $M \oplus R^n \cong N \oplus R^n$ for some $n \in \mathbb{N}_0$.*

Proof: "if" is clear. For the other direction, note that, since short exact sequences of projective modules split, Proposition 2.6 says that $[M] = [N]$ implies the existence of a finitely generated projective module L such that $M \oplus L \cong N \oplus L$. Adding a finitely generated projective module Q such that $L \oplus Q \cong R^n$ for some $n \in \mathbb{N}_0$ (this exists; see, for example, [Mag02, Proposition 2.21]), we therefore get $M \oplus R^n \cong N \oplus R^n$ as desired. □

A basic result that is used repeatedly is

Proposition 2.8. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}^R_\square(\mathrm{f})$ containing the zero complex, and suppose that X is a complex in $\mathcal{C}$ such that the shifted complex ΣX and the mapping cone $\mathcal{M}(\mathbb{1}_X)$ are in $\mathcal{C}$. Then $[\Sigma X] = -[X]$ in $K_0(\mathcal{C})$.*

Proof: Because of the assumptions, the exact sequence $0 \to X \to \mathcal{M}(\mathbb{1}_X) \to \Sigma X \to 0$ from Theorem 0.2 is in $\mathcal{C}$, implying that $[\mathcal{M}(\mathbb{1}_X)] = [X] + [\Sigma X]$ in $K_0(\mathcal{C})$. Since the identity map $\mathbb{1}_X$ is a homology isomorphism, its mapping cone $\mathcal{M}(\mathbb{1}_X)$ is exact according to Theorem 0.4, and hence $[\mathcal{M}(\mathbb{1}_X)] = 0$ and we conclude that $[\Sigma X] = -[X]$. □

When Proposition 2.8 is applied to the complex $\Sigma^{-1}X$, the complex X shifted one degree to the right, one obtains that $[\Sigma^{-1}X] = -[X]$. In general, if our category $\mathcal{C}$ is large enough that it contains all the necessary complexes, $[\Sigma^n X] = (-1)^n[X]$ in $K_0(\mathcal{C})$ for all $n \in \mathbb{Z}$.

Proposition 2.8 can be taken one step further.

Proposition 2.9. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}^R_\square(\mathrm{f})$ containing the zero complex, and suppose that $\phi\colon X \to Y$ is a homology isomorphism in $\mathcal{C}$ such that the shifted complex ΣX and the mapping cones $\mathcal{M}(\mathbb{1}_X)$ and $\mathcal{M}(\phi)$ are in $\mathcal{C}$. Then $[X] = [Y]$ in $K_0(\mathcal{C})$.*

Proof: According to Theorem 0.4, it follows from ϕ being a homology isomorphism that $\mathcal{M}(\phi)$ is exact, and hence that $[\mathcal{M}(\phi)] = 0$. From the exact sequence $0 \to Y \to \mathcal{M}(\phi) \to \Sigma X \to 0$ (see Theorem 0.2) and Proposition 2.8 we now get $[Y] = [\mathcal{M}(\phi)] - [\Sigma X] = [X]$. □

In this thesis we shall consider several homomorphisms between Grothendieck groups. Among these are the homomorphisms $\mathcal{A}$, $\mathcal{H}$, $\mathcal{R}$ and χ described in Theorems 2.10 through 2.13 below. The first theorem shows how we can switch from complexes to modules at the level of Grothendieck groups in certain circumstances by taking the alternating sum of the modules in a complex.

Theorem 2.10. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}^R_\square(\mathrm{f})$ containing the zero complex, and let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}^R_0(\mathrm{f})$ containing all kernels of its homomorphism (and therefore also containing the zero module). Suppose that for any X in $\mathcal{C}$, the modules X_ℓ are in $\mathcal{C}_0$. Then there is a group homomorphism $\mathcal{A}\colon K_0(\mathcal{C}) \to K_0(\mathcal{C}_0)$ given by $\mathcal{A}([X]) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell]$.*

Proof: The assumptions ensure that, for X in $\mathcal{C}$, $a(X) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell]$ is a well-defined element of $K_0(\mathcal{C}_0)$, so it only remains to verify that the relations in $K_0(\mathcal{C})$ are preserved under a, so that a induces the homomorphism $\mathcal{A}$.

So suppose that X is exact, and let $Z_\ell = \ker\partial^X_\ell = \operatorname{im}\partial^X_{\ell+1}$. For each $\ell \in \mathbb{Z}$, we then have an exact sequence $0 \to Z_\ell \to X_\ell \to Z_{\ell-1} \to 0$, which, according to the assumption, is in $\mathcal{C}_0$. It follows that

$$a(X) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell] = \sum_{\ell\in\mathbb{Z}}(-1)^\ell([Z_\ell] + [Z_{\ell-1}]) = 0.$$

Suppose next that $0 \to U \to V \to W \to 0$ is an exact sequence in $\mathcal{C}$. There is then an exact sequence $0 \to U_\ell \to V_\ell \to W_\ell \to 0$ of modules in each degree $\ell \in \mathbb{Z}$, proving that

$$a(V) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[V_\ell] = \sum_{\ell\in\mathbb{Z}}(-1)^\ell([U_\ell] + [W_\ell]) = a(U) + a(W).$$

This completes the proof that a preserves the relations in $K_0(\mathcal{C})$ and thereby induces the homomorphism $\mathcal{A}$. □

In many other cases we can switch from complexes to modules at the level of Grothendieck groups by taking the alternating sum of the *homology* modules of a complex.

Theorem 2.11. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}^R_\square(\mathrm{f})$ containing the zero complex, and let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}^R_0(\mathrm{f})$ containing the homology modules $\mathrm{H}_\ell(X)$ of all X in $\mathcal{C}$ (and hence also containing the zero module). Assume either that all homologies $\mathrm{H}(X)$ of complexes X in $\mathcal{C}$ are concentrated in degree 0, or that $\mathcal{C}_0$ contains all the kernels of its homomorphisms. Then, in either case, there is group homomorphism $\mathcal{H}\colon K_0(\mathcal{C}) \to K_0(\mathcal{C}_0)$ given by $\mathcal{H}([X]) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[\mathrm{H}_\ell(X)]$.*

Proof: The assumptions ensure that, for X in $\mathcal{C}$, $h(X) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[\mathrm{H}_\ell(X)]$ is a well-defined element of $K_0(\mathcal{C}_0)$, so it only remains to verify that the relations in $K_0(\mathcal{C})$ are preserved under h, so that h induces the homomorphism $\mathcal{H}$.

If X is exact, then clearly $h(X) = 0$. Now let $0 \to U \to V \to W \to 0$ be an exact sequence of complexes in $\mathcal{C}$, and consider the long exact sequence on homology:

$$\cdots \to \mathrm{H}_\ell(U) \to \mathrm{H}_\ell(V) \to \mathrm{H}_\ell(W) \to \mathrm{H}_{\ell-1}(U) \to \cdots .$$

If we assume that all homologies are concentrated in degree 0, the above is simply the exact sequence $0 \to \mathrm{H}_0(U) \to \mathrm{H}_0(V) \to \mathrm{H}_0(W) \to 0$, proving that $h(V) = h(U) + h(W)$ in $K_0(\mathcal{C}_0)$. So assume instead that $\mathcal{C}_0$ contains all the kernels of its homomorphisms. For $\ell \in \mathbb{Z}$, let Z^U_ℓ, Z^V_ℓ and Z^W_ℓ denote the kernels of the maps $\mathrm{H}_\ell(U) \to \mathrm{H}_\ell(V)$, $\mathrm{H}_\ell(V) \to \mathrm{H}_\ell(W)$ and $\mathrm{H}_\ell(W) \to \mathrm{H}_{\ell-1}(U)$, respectively. We then have exact sequences

$$\begin{aligned} 0 \to Z^U_\ell \to \mathrm{H}_\ell(U) \to Z^V_\ell&, \\ 0 \to Z^V_\ell \to \mathrm{H}_\ell(V) \to Z^W_\ell& \quad\text{and} \\ 0 \to Z^W_\ell \to \mathrm{H}_\ell(W) \to Z^U_{\ell-1}&, \end{aligned}$$

which, according to the assumption, all are in $\mathcal{C}_0$. We now see that

$$\begin{aligned} h(V) &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell[\mathrm{H}_\ell(V)] \\ &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell([Z_\ell^V]+[Z_\ell^W]) \\ &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell([Z_\ell^U]+[Z_\ell^V]) + \sum_{\ell\in\mathbb{Z}}(-1)^\ell([Z_\ell^W]+[Z_{\ell-1}^U]) \\ &= h(U)+h(W). \end{aligned}$$

This completes the proof that h preserves the relations in $K_0(\mathcal{C})$ and thereby induces the homomorphism $\mathcal{H}$. □

Sometimes we can go the other way, switching from modules to complexes, by taking projective resolutions.

Theorem 2.12. *Let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}_0^R(\mathrm{f,pd})$ containing the zero module, and let $\mathcal{C}$ be a full subcategory of $\mathcal{C}_\square^R(\mathrm{f})$ containing the zero complex and for each exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{C}_0$ containing projective resolutions U, V and W of L, M and N respectively, such that there is an exact sequence $0 \to U \to V \to W \to 0$ in $\mathcal{C}$. Suppose that $\mathcal{C}$ is closed under the shift operator $\Sigma(-)$ and under the formation of mapping cones $\mathcal{M}(-)$. Then there is a group homomorphism $\mathcal{R}\colon K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ given by $\mathcal{R}([M]) = [X]$, where $X \in \mathcal{C}$ is a projective resolution of M.*

Proof: If M is a module in $\mathcal{C}_0$, then the assumptions ensure that there is at least one projective resolution X of M in $\mathcal{C}$. We first verify that $[X] = [X']$ whenever $X, X' \in \mathcal{C}$ are both finite projective resolutions of M. In this case, there is a homology isomorphism $X \xrightarrow{\simeq} X'$ (see, for example, [HS97, Proposition IV.4.3]), so according to Proposition 2.9 and the assumptions made on $\mathcal{C}$, $[X] = [X']$. Thus, it makes sense to define, for any $M \in \mathcal{C}_0$, an element $r(M)$ in $K_0(\mathcal{C})$, given by $r(M) = [X]$ for any choice $X \in \mathcal{C}$ of projective resolution of M. To show that r in fact induces a homomorphism $\mathcal{R}\colon K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$, it suffices to demonstrate that if $0 \to L \to M \to N \to 0$ is a short exact sequence in $\mathcal{C}_0$, then projective resolutions U, V and W of L, M and N, respectively, exist, such that there is a short exact sequence $0 \to U \to V \to W \to 0$ in $\mathcal{C}$. But this is exactly what we have assumed. □

Although the requirement that short exact sequences of projective resolutions of short exact sequences of modules should exist sounds slightly awkward, it is not so rarely satisfied. Indeed, if $0 \to L \to M \to N \to 0$ is a short exact sequence of modules and U and W are projective resolutions of L and N respectively, then there is a projective resolution V of M whose modules V_ℓ are the direct sums $U_\ell \oplus W_\ell$, such that the (degreewise) inclusion $U \hookrightarrow V$ and the (degreewise) projection $V \twoheadrightarrow W$ are both morphisms of complexes (see, for example, [Mag02, (3.51)]). Consequently, the requirement is satisfied by all categories of complexes that are closed under this construction.

In the case that $\mathcal{C}_0$, $\mathcal{C}_0'$ and $\mathcal{C}$ are categories such that the homomorphisms $\mathcal{R}\colon K_0(\mathcal{C}_0) \to K_0(\mathcal{C})$ and $\mathcal{A}\colon K_0(\mathcal{C}) \to K_0(\mathcal{C}_0')$ from Theorems 2.12 and 2.10, respectively, are defined, the composition $\chi = \mathcal{A} \circ \mathcal{R}$ is a homomorphism $K_0(\mathcal{C}_0) \to K_0(\mathcal{C}_0')$ given by $\chi([M]) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell]$ for a finite projective resolution X of M. One often skips the intermediate step and defines χ directly as a homomorphism $K_0(\mathcal{C}_0) \to K_0(\mathcal{C}_0')$.

Theorem 2.13. *Let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}_0^R(\text{f,pd})$ containing the zero module, and let $\mathcal{C}_0'$ be a full subcategory of $\mathcal{C}_0^R(\text{f})$ containing $\mathcal{C}_0^R(\text{f,P})$. Suppose that every module in $\mathcal{C}_0$ has a finite projective resolution. Then there is a group homomorphism $\chi\colon K_0(\mathcal{C}_0) \to K_0(\mathcal{C}_0')$ given by $\chi([M]) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell]$, where X is a bounded finite projective resolution of M.*

Proof: If M is a module in $\mathcal{C}_0$, the assumptions ensure that we can find a bounded finite projective resolution X of M and that $\sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell]$ is a well-defined element of $\mathcal{C}_0'$. Suppose that X and X' are both bounded finite projective resolutions of M. We then have (see, for example, [Mag02, Corollary 3.47])

$$\Big(\coprod_{\ell \text{ even}} X_\ell\Big) \oplus \Big(\coprod_{\ell \text{ odd}} X_\ell'\Big) \cong \Big(\coprod_{\ell \text{ odd}} X_\ell\Big) \oplus \Big(\coprod_{\ell \text{ even}} X_\ell'\Big),$$

proving that $\sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell] = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell']$ in $K_0(\mathcal{C}_0')$. Thus, we can associate to every element $M \in \mathcal{C}_0$ the element $\sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell]$ in $K_0(\mathcal{C}_0')$ for any choice of bounded finite projective resolution X of M. The fact that this association induces a homomorphism χ follows as in the proof of Theorem 2.12 (and [Mag02] (3.51)), since we are free to construct short exact sequences of projective resolutions of short exact sequences of modules. □

A homomorphism χ as in Theorem 2.13 is called an *Euler characteristic*. This term is natural since the usual Euler characteristic $\chi^R(-)$ is defined when R is Noetherian and local, in which case $K_0(R) = K_0(\mathcal{C}_0^R(\text{f,P}))$ is isomorphic to $\mathbb{Z}$ (see Example 2.14 below), by an isomorphism taking $\chi([M]) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell[X_\ell]$ to $\sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{rank}_R X_\ell = \chi^R(M)$. Thus, the usual Euler characteristic is an application of the above theorem to the case where $\mathcal{C}_0 = \mathcal{C}_0^R(\text{f,pd})$ and $\mathcal{C}_0' = \mathcal{C}_0^R(\text{f,P})$.

If $\mathcal{C}_0$ is a full subcategory of $\mathcal{C}_0^R(\text{f})$, the Grothendieck group $K_0(\mathcal{C}_0)$ has two fundamental properties: (i) it is an Abelian group and (ii) there is a function $\pi\colon \mathcal{I}(\mathcal{C}_0) \to K_0(\mathcal{C}_0)$ (given by $\pi(\mathrm{c}(M)) = [M]$) that is additive on short exact sequences. The definition of Grothendieck groups implies that $K_0(\mathcal{C}_0)$ is universal with respect to these properties, in the sense that if A is any other Abelian group equipped with a function $r\colon \mathcal{I}(\mathcal{C}_0) \to A$ that is additive on short exact sequences, then r factors uniquely through $K_0(\mathcal{C}_0)$: that is, a unique homomorphism $\bar{r}\colon K_0(\mathcal{C}_0) \to A$ exists such that $r = \bar{r}\pi$. Henceforth, when given such a function r, we shall also denote the induced function by r.

The universal property of Grothendieck groups often enables us to calculate the Grothendieck group of a category. A few examples of this are listed here.

Example 2.14. The usual free rank is a function $\mathcal{I}(\mathcal{C}_0^R(\text{f,F})) \to \mathbb{Z}$, which is additive on short exact sequences. Since rank $R = 1$, the induced homomorphism $G_0^R(\text{f,F}) \to \mathbb{Z}$ is surjective. To see that it is in fact also injective, recall that, since $\mathcal{C}_0^R(\text{f,F})$ is closed under direct sum, all elements of $G_0^R(\text{f,F})$ can be written in the form $[M] - [N]$. If such an element is in the kernel of rank, then M and N must have the same free rank: that is, $M \cong R^n \cong N$ for some $n \in \mathbb{N}_0$. Consequently $[M] = [N]$, so rank is injective as a map $G_0^R(\text{f,F}) \to \mathbb{Z}$. We have therefore shown that $G_0^R(\text{f,F}) \cong \mathbb{Z}$. In particular, if R is local, then any finitely generated projective module is free, so $\mathcal{C}_0^R(\text{f,P})$ is the same as $\mathcal{C}_0^R(\text{f,F})$, and it follows that $K_0(R) \cong \mathbb{Z}$. Note that, regardless of whether R is local or not, the map $\mathbb{Z} \to K_0(R)$ given by $1 \mapsto [R]$ is always injective, since $[R^m] = [R^n]$ according to Corollary 2.7 implies that $m = n$.

Example 2.15. Consider the category $\mathcal{C}_0^{\mathbb{Z}}(\text{l})$ of finite-length $\mathbb{Z}$-modules; this is exactly the same as the category of finite Abelian groups. The order of finite groups is a function $|-|\colon \mathcal{I}(\mathcal{C}_0^{\mathbb{Z}}(\text{l})) \to \mathbb{Q}^+$, which is multiplicative on short exact sequences. The induced homomorphism $|-|\colon G_0^{\mathbb{Z}}(\text{l}) \to \mathbb{Q}^+$, which is given by $[H] - [K] \mapsto |H|/|K|$, is surjective since Abelian groups of every positive order exist. To see that $|-|$ is also injective, we can assume, as in the previous example, that H and K are finitely generated Abelian groups with $|H| = |K|$. Now, recall that H has a composition series

$$H = H_0 \rhd H_1 \rhd \cdots \rhd H_{t-1} \rhd H_t = \{0_H\},$$

in which each quotient H_{i-1}/H_i is cyclic of prime order: that is, $H_{i-1}/H_i \cong \mathbb{Z}/p_i\mathbb{Z}$ for some prime number p_i. From the exact sequences $0 \to H_i \to H_{i-1} \to \mathbb{Z}/p_i\mathbb{Z} \to 0$ we therefore get $[H] = \sum_{i=1}^t [\mathbb{Z}/p_i\mathbb{Z}]$ in $G_0^{\mathbb{Z}}(\text{l})$. Similarly, we can reduce $[K]$ into composition factors, so that $[K] = \sum_{j=1}^s [\mathbb{Z}/q_j\mathbb{Z}]$ where the q_j's are prime numbers. Based on the assumption that $|H| = |K|$, it now follows that $\prod_{i=1}^t p_i = \prod_{j=1}^s q_j$, and the fundamental theorem of arithmetics implies that $t = s$ and that, after rearrangement of the q_j's, $p_i = q_i$ for all i. Consequently $[H] = \sum_{i=1}^t [\mathbb{Z}/p_i\mathbb{Z}] = [K]$ in $G_0^{\mathbb{Z}}(\text{l})$, so $|-|$ is injective. We have therefore proved that $G_0^{\mathbb{Z}}(\text{l}) \cong \mathbb{Q}^+$.

Example 2.16. Suppose that R is local with maximal ideal $\mathfrak{m}$ and quotient field $k = R/\mathfrak{m}$, and consider the category $\mathcal{C}_0^R(\text{l})$ of modules of finite length. The length of modules is then a function $\text{length}_R\colon \mathcal{I}(\mathcal{C}_0^R(\text{l})) \to \mathbb{Z}$, which is additive on short exact sequences. The induced homomorphism $\text{length}_R\colon G_0^R(\text{l}) \to \mathbb{Z}$ is surjective since $\operatorname{length}_R k = 1$. To demonstrate that length_R is injective, we can assume, as in the previous examples, that M and N are modules with $\operatorname{length}_R M = \operatorname{length}_R N < \infty$. Now, note that in a composition series

$$M = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0,$$

where $t = \operatorname{length}_R M$, each quotient M_{i-1}/M_i is isomorphic to k. From the exact sequences $0 \to M_i \to M_{i-1} \to k \to 0$, we therefore inductively see that $[M] = \sum_{i=1}^{t}[k] = (\operatorname{length}_R M)[k]$. From the assumption that $\operatorname{length}_R M = \operatorname{length}_R N$, it now follows that $[M] = (\operatorname{length}_R M)[k] = [N]$ in $G_0^R(\mathrm{l})$, so length_R is indeed injective. Hence we have proved that $G_0^R(\mathrm{l}) \cong \mathbb{Z}$.

This section concludes by investigating how group homomorphisms between Grothendieck groups can be induced from functors between the underlying categories. Suppose R and R' are (nontrivial, unitary and commutative) rings and that $\mathcal{C}$ and $\mathcal{C}'$ are full subcategories of $\mathcal{C}_\Box^R(\mathrm{f})$ and $\mathcal{C}_\Box^{R'}(\mathrm{f})$, respectively. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a covariant, exact functor taking exact complexes to exact complexes. Then there is an induced group homomorphism $K_0(F)\colon K_0(\mathcal{C}) \to K_0(\mathcal{C}')$ given by $K_0(F)([X]) = [F(X)]$ for $X \in \mathcal{C}$; the relations in $K_0(\mathcal{C})$ are preserved because of the assumption that F is exact and takes exact complexes to exact complexes.

In case $\mathcal{C}$ is a subcategory of $\mathcal{C}_0^R(\mathrm{f})$, the requirement that F must take exact complexes to exact complexes is superfluous, and the only remaining requirement is that F be exact. When the categories $\mathcal{C}_0^R(\mathrm{f,P})$ and $\mathcal{C}_0^{R'}(\mathrm{f,P})$ from which we form the traditional K_0-groups $K_0(R)$ and $K_0(R')$ are considered, it suffices in some cases that F is additive.

Theorem 2.17. *If R and R' are (nontrivial, unitary and commutative) rings and $F\colon \mathcal{C}_0^R \to \mathcal{C}_0^{R'}$ is an additive functor such that $F(R) \in \mathcal{C}_0^{R'}(\mathrm{f,P})$, then F restricts to an exact functor $\mathcal{C}_0^R(\mathrm{f,P}) \to \mathcal{C}_0^{R'}(\mathrm{f,P})$. Consequently, F induces a homomorphism $K_0(F)\colon K_0(R) \to K_0(R')$.*

Proof: If $P \in \mathcal{C}_0^R(\mathrm{f,P})$, then $Q \in \mathcal{C}_0^R(\mathrm{f,P})$ exists such that $P \oplus Q \cong R^n$ for some $n \in \mathbb{N}_0$. Since F is additive,

$$F(P) \oplus F(Q) \cong F(P \oplus Q) \cong F(R^n) \cong F(R)^n,$$

which, by assumption, is a module in $\mathcal{C}_0^{R'}(\mathrm{f,P})$. Consequently, $F(P)$ is a direct summand of a finitely generated projective R'-module, and hence it is itself a finitely generated projective R'-module. This proves that F restricts to a functor $\mathcal{C}_0^R(\mathrm{f,P}) \to \mathcal{C}_0^{R'}(\mathrm{f,P})$. Since all exact sequences in $\mathcal{C}_0^R(\mathrm{f,P})$ split, the additivity of F ensures that this restriction of F is exact. It follows that F induces a homomorphism $K_0(F)\colon K_0(R) \to K_0(R')$. $\square$

An important application of Theorem 2.17 is in the case of a ring homomorphism $\rho\colon R \to R'$. We can then consider R' as an (R', R)-bimodule, and we get an induced functor $R' \otimes_R -\colon \mathcal{C}_0^R \to \mathcal{C}_0^{R'}$ that satisfies the conditions of Theorem 2.17 since it is additive and takes R to $R' \otimes_R R \cong R' \in \mathcal{C}_0^{R'}(\text{f,P})$.

Definition 2.18. If R and R' are (nontrivial, unitary and commutative) rings and $\rho\colon R \to R'$ is a ring homomorphism, then we denote by $K_0(\rho)$ the functor $K_0(R) \to K_0(R')$ induced by the functor $R' \otimes_R -\colon \mathcal{C}_0^R \to \mathcal{C}_0^{R'}$ in the sense of Theorem 2.17.

Chapter 3

Grothendieck group isomorphisms

This chapter establishes a number of surprising isomorphisms between related Grothendieck groups. The results obtained are all consequences of the following theorem, which is therefore referred to as the main theorem.

Main Theorem. *Suppose that d is a nonnegative integer and $S = (S_1, \dots, S_d)$ is a d-tuple of multiplicative systems of R. Then the group homomorphism*

$$\iota\colon G_d^R(\text{f,P}|S\text{-tor}) \to G_\square^R(\text{f,P}|S\text{-tor})$$

given by $\iota([X]) = [X]$ is an isomorphism.

Note that the two $[X]$'s are different: one is a member of $G_d^R(\text{f,P}|S\text{-tor})$, whereas the other is a member of $G_\square^R(\text{f,P}|S\text{-tor})$. Note also that the map ι is not trivially injective, although it is induced by the inclusion of $\mathcal{C}_d^R(\text{f,P}|S\text{-tor})$ in $\mathcal{C}_\square^R(\text{f,P}|S\text{-tor})$. The fact that $\mathcal{C}_d^R(\text{f,P}|S\text{-tor})$ is a subcategory of $\mathcal{C}_\square^R(\text{f,P}|S\text{-tor})$, however, does not help anything but to ensure that ι is well defined.

In the case that $d = 0$, the requirement of being homologically S-torsion is eliminated, and the main theorem states that $K_0(R) = G_0^R(\text{f,P}) \cong G_\square^R(\text{f,P})$.

Establishing the main theorem is a cumbersome task. We will construct an inverse to ι as follows. Given a complex $Y \in \mathcal{C}_\square^R(\text{f,P}|S\text{-tor})$, we choose $n \in \mathbb{Z}$ so that the shifted complex $\Sigma^n Y$ is in $\mathcal{C}_e^R(\text{f,P}|S\text{-tor})$ for some $e > d$. To this complex we associate an element $w_e(\Sigma^n Y) \in G_{e-1}^R(\text{f,P}|S\text{-tor})$. Repeating this process a finite number of times, we end up with an element $w_{d+1} \cdots w_e(\Sigma^n Y)$ in $G_d^R(\text{f,P}|S\text{-tor})$. This is the image of $[Y]$ under the inverse of ι.

We begin by investigating the concepts of *contractions* and *Koszul complexes*.

3.1 Contractions

Throughout this section, d denotes a positive integer and $S = (S_1, \dots, S_d)$ denotes a d-tuple of multiplicative systems of R.

Definition 3.1. Let $X \in \mathcal{C}^R$ be a complex. A d-tuple $\alpha = (\alpha^1, \dots, \alpha^d)$ of families $\alpha^\nu = (\alpha^\nu_\ell)_{\ell \in \mathbb{Z}}$ of homomorphisms $\alpha^\nu_\ell\colon X_\ell \to X_{\ell+1}$ is an *S-contraction of X with weight $s = (s_1, \dots, s_d) \in S_1 \times \cdots \times S_d$* if

$$\partial^X_{\ell+1}\alpha^\nu_\ell + \alpha^\nu_{\ell-1}\partial^X_\ell = s_\nu \mathbb{1}_{X_\ell}$$

for all $\ell \in \mathbb{Z}$ and $\nu = 1, \ldots, d$.

The situation is as follows.

$$\cdots \rightleftarrows X_{\ell+1} \underset{\alpha_\ell^\nu}{\overset{\partial_{\ell+1}^X}{\rightleftarrows}} X_\ell \underset{\alpha_{\ell-1}^\nu}{\overset{\partial_\ell^X}{\rightleftarrows}} X_{\ell-1} \rightleftarrows \cdots .$$

Note that the existence of an S-contraction of X with weight $s = (s_1, \ldots, s_d)$ is equivalent to the condition that the morphisms $s_\nu \mathbb{1}_X \colon X \to X$ for $\nu = 1, \ldots, d$ are null-homotopic.

Proposition 3.2. *Each complex $X \in \mathcal{C}_\square^R(\text{f,P}|S\text{-tor})$ has an S-contraction.*

Proof: For each ν the $S_\nu^{-1}R$-complex $S_\nu^{-1}X$ is exact, bounded and consists of finitely generated projective $S_\nu^{-1}R$-modules, so the identity morphism $\mathbb{1}_{S_\nu^{-1}X}$ on $S_\nu^{-1}X$ is null-homotopic (see, for example, [HS97, Theorem IV.4.1]). Thus, we can find $S_\nu^{-1}R$-homomorphisms $b_\ell^\nu \colon S_\nu^{-1}X_\ell \to S_\nu^{-1}X_{\ell+1}$ such that

$$\partial_{\ell+1}^{S_\nu^{-1}X} b_\ell^\nu + b_{\ell-1}^\nu \partial_\ell^{S_\nu^{-1}X} = \mathbb{1}_{S_\nu^{-1}X_\ell}$$

for all $\ell \in \mathbb{Z}$.

Because of the natural isomorphism (see, for example, [Eis95, Proposition 2.10])

$$\operatorname{Hom}_{S_\nu^{-1}R}(S_\nu^{-1}X_\ell, S_\nu^{-1}X_{\ell+1}) \cong S_\nu^{-1} \operatorname{Hom}_R(X_\ell, X_{\ell+1})$$

and because our complex is bounded, the $S_\nu^{-1}R$-homomorphism b_ℓ^ν can be written as β_ℓ^ν / t_ν for an R-homomorphism $\beta_\ell^\nu \colon X_\ell \to X_{\ell+1}$ and some common denominator $t_\nu \in S_\nu$. For any $x \in X_\ell$, we must now have in $S_\nu^{-1}X_\ell$ that

$$(\partial_{\ell+1}^X \beta_\ell^\nu + \beta_{\ell-1}^\nu \partial_\ell^X)(x)/t_\nu = x/1.$$

Consequently, we can find $u_{\nu,x} \in S_\nu$ depending on x so that in X_ℓ,

$$u_{\nu,x}(\partial_{\ell+1}^X \beta_\ell^\nu + \beta_{\ell-1}^\nu \partial_\ell^X)(x) = u_{\nu,x} t_\nu x.$$

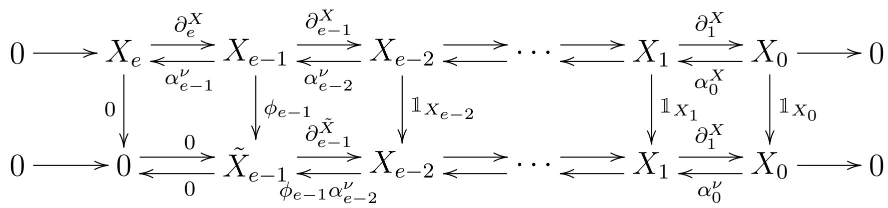
Since X is bounded and consists of finitely generated modules, by multiplying a finite number of $u_{\nu,x}$'s, we can obtain an element $u_\nu \in S_\nu$, independent of x and of ℓ, such that $u_\nu(\partial_{\ell+1}^X \beta_\ell^\nu + \beta_{\ell-1}^\nu \partial_\ell^X)(x) = u_\nu t_\nu x$ for all $\ell \in \mathbb{Z}$ and all $x \in X_\ell$. Setting $\alpha_\ell^\nu = u_\nu \beta_\ell^\nu$ and $s_\nu = u_\nu t_\nu$, we see that $\alpha = (\alpha^1, \ldots, \alpha^d)$, where $\alpha^\nu = (\alpha_\ell^\nu)_{\ell \in \mathbb{Z}}$, is an S-contraction of X with weight $s = (s_1, \ldots, s_d)$. □

Definition 3.3. Let X and Y be complexes in $\mathcal{C}^R$ with S-contractions α and β, respectively, and let $\phi \colon X \to Y$ be a morphism of complexes. Then α and β are said to be *compatible with* ϕ if they have the same weight and $\phi_{\ell+1}\alpha_\ell^\nu = \beta_\ell^\nu \phi_\ell$ for all $\ell \in \mathbb{Z}$ and $\nu = 1, \ldots, d$.

Theorem 3.4 below provides an example of a situation where an S-contraction on a complex induces an S-contraction on another complex. Although the hypotheses of the theorem are very specific, the theorem turns out to be applicable in several situations.

Theorem 3.4. *Let X be a complex in $\mathcal{C}_e^R$, where $e > 1$, and suppose that α is an S-contraction of X with weight s. Let $\tilde{X}$ be another complex in $\mathcal{C}_e^R$, and suppose that the complex $\tilde{X}$ is identical to X except for the modules and differentials in degrees e and $e-1$. Suppose further that $\tilde{X}_e = 0$ and that a morphism $\phi\colon X \to \tilde{X}$ exists such that $\phi_\ell = \mathbb{1}_{X_\ell}$ for $\ell = 0, \dots, e-2$ and such that ϕ_{e-1} is surjective. Then the S-contraction α on X induces an S-contraction $\tilde{\alpha}$ on $\tilde{X}$ with weight s such that α and $\tilde{\alpha}$ are compatible with the morphism ϕ; for $\nu = 1, \dots, d$, $\tilde{\alpha}^\nu$ is defined by setting $\tilde{\alpha}^\nu_{e-2} = \phi_{e-1}\alpha^\nu_{e-2}$ and $\tilde{\alpha}^\nu_\ell = \alpha^\nu_\ell$ for $\ell = 0, \dots, e-3$.*

Proof: The situation is as follows.



To demonstrate that $\tilde{\alpha}$ is an S-contraction of $\tilde{X}$ with weight s, we need to verify that $s_\nu \mathbb{1}_{\tilde{X}_{e-1}} = \phi_{e-1}\alpha^\nu_{e-2}\partial^{\tilde{X}}_{e-1}$ and that $s_\nu \mathbb{1}_{X_{e-2}} = \partial^{\tilde{X}}_{e-1}\phi_{e-1}\alpha^\nu_{e-2} + \alpha^\nu_{e-3}\partial^X_{e-2}$ for $\nu = 1, \dots, d$. Since ϕ_{e-1} is surjective, the first of these equations follows from the following calculation.

$$\begin{aligned} \phi_{e-1}\alpha^\nu_{e-2}\partial^{\tilde{X}}_{e-1}\phi_{e-1} &= \phi_{e-1}\alpha^\nu_{e-2}\partial^X_{e-1} \\ &= \phi_{e-1}(s_\nu \mathbb{1}_{X_{e-1}} - \partial^X_e \alpha^\nu_{e-1}) \\ &= s_\nu \phi_{e-1}. \end{aligned}$$

The second equation is even easier:

$$\partial^{\tilde{X}}_{e-1}\phi_{e-1}\alpha^\nu_{e-2} + \alpha^\nu_{e-3}\partial^X_{e-2} = \partial^X_{e-1}\alpha^\nu_{e-2} + \alpha^\nu_{e-3}\partial^X_{e-2} = s_\nu \mathbb{1}_{X_{e-2}}.$$

By construction of $\tilde{\alpha}$, α and $\tilde{\alpha}$ are compatible with the morphism ϕ. Thus the theorem has been proven. □

Given an S-contraction α of X with weight $s = (s_1, \dots, s_d)$ and a d-tuple $t = (t_1, \dots, t_d) \in S_1 \times \cdots \times S_d$, we can construct an S-contraction $t\alpha$ of X with weight $st = (s_1t_1, \dots, s_dt_d)$ by setting $t\alpha = (t_1\alpha^1, \dots, t_d\alpha^d)$ where $t_\nu\alpha^\nu = (t_\nu\alpha^\nu_\ell)_{\ell\in\mathbb{Z}}$. We can also shift α n degrees to the left for some $n \in \mathbb{Z}$ to form an S-contraction $\Sigma^n\alpha$ of $\Sigma^n X$ with weight s by setting $\Sigma^n\alpha = (\Sigma^n\alpha^1, \dots, \Sigma^n\alpha^d)$ where $\Sigma^n\alpha^\nu = ((-1)^n\alpha^\nu_{\ell-n})_{\ell\in\mathbb{Z}}$.

The following theorem shows how to construct a natural S-contraction of the mapping cone of a morphism between two complexes that both have S-contractions.

Theorem 3.5. *Let $\phi\colon X \to Y$ be a morphism of complexes and let α and β be S-contractions of X and Y, respectively, with weights s and t, respectively. Define for $\nu = 1, \dots, d$ and $\ell \in \mathbb{Z}$ the homomorphism*

$$(\beta * \alpha)^\nu_\ell = \begin{pmatrix} s_\nu \beta^\nu_\ell & \beta^\nu_\ell \phi_\ell \alpha^\nu_{\ell-1} \\ 0 & -t_\nu \alpha^\nu_{\ell-1} \end{pmatrix} : \mathcal{M}(\phi)_\ell = \begin{matrix} Y_\ell \\ \oplus \\ X_{\ell-1} \end{matrix} \to \begin{matrix} Y_{\ell+1} \\ \oplus \\ X_\ell \end{matrix} = \mathcal{M}(\phi)_{\ell+1}.$$

Then $(\beta * \alpha) = ((\beta * \alpha)^1, \dots, (\beta * \alpha)^d)$, where $(\beta * \alpha)^\nu = ((\beta * \alpha)^\nu_\ell)_{\ell \in \mathbb{Z}}$, is an S-contraction of the mapping cone $\mathcal{M}(\phi)$ of ϕ with weight $st = (s_1 t_1, \dots, s_d t_d)$, and the S-contractions $s\beta$, $(\beta * \alpha)$ and $\Sigma t\alpha$ are compatible with the morphisms in the canonical exact sequence

$$0 \to Y \to \mathcal{M}(\phi) \to \Sigma X \to 0. \tag{3.1}$$

Proof: This is just a matter of verification. First we need to show that

$$\partial^{\mathcal{M}(\phi)}_{\ell+1} (\beta * \alpha)^\nu_\ell + (\beta * \alpha)^\nu_{\ell-1} \partial^{\mathcal{M}(\phi)}_\ell = s_\nu t_\nu \mathbb{1}_{\mathcal{M}(\phi)_\ell}$$

for $\nu = 1, \dots, d$ and $\ell \in \mathbb{Z}$. This comes down to the following multiplication of matrices.

$$\begin{aligned} &\begin{pmatrix} \partial^Y_{\ell+1} & \phi_\ell \\ 0 & -\partial^X_\ell \end{pmatrix} \begin{pmatrix} s_\nu \beta^\nu_\ell & \beta^\nu_\ell \phi_\ell \alpha^\nu_{\ell-1} \\ 0 & -t_\nu \alpha^\nu_{\ell-1} \end{pmatrix} + \begin{pmatrix} s_\nu \beta^\nu_{\ell-1} & \beta^\nu_{\ell-1} \phi_{\ell-1} \alpha^\nu_{\ell-2} \\ 0 & -t_\nu \alpha^\nu_{\ell-2} \end{pmatrix} \begin{pmatrix} \partial^Y_\ell & \phi_{\ell-1} \\ 0 & -\partial^X_{\ell-1} \end{pmatrix} \\ &= \begin{pmatrix} s_\nu(\partial^Y_{\ell+1}\beta^\nu_\ell + \beta^\nu_{\ell-1}\partial^Y_\ell) & (\partial^Y_{\ell+1}\beta^\nu_\ell - t_\nu)\phi_\ell \alpha^\nu_{\ell-1} + \beta^\nu_{\ell-1}\phi_{\ell-1}(s_\nu - \alpha^\nu_{\ell-2}\partial^X_{\ell-1}) \\ 0 & t_\nu(\partial^X_\ell \alpha^\nu_{\ell-1} + \alpha^\nu_{\ell-2}\partial^X_{\ell-1}) \end{pmatrix} \\ &= \begin{pmatrix} s_\nu t_\nu \mathbb{1}_{Y_\ell} & \beta^\nu_{\ell-1}(-\partial^Y_\ell \phi_\ell + \phi_{\ell-1}\partial^X_\ell)\alpha^\nu_{\ell-1} \\ 0 & s_\nu t_\nu \mathbb{1}_{X_{\ell-1}} \end{pmatrix} \\ &= \begin{pmatrix} s_\nu t_\nu \mathbb{1}_{Y_\ell} & 0 \\ 0 & s_\nu t_\nu \mathbb{1}_{X_{\ell-1}} \end{pmatrix}. \end{aligned}$$

Here the second equality follows from α and β being contractions with weights s and t, respectively, and the third equality follows from ϕ being a morphism of complexes. Consequently, $(\beta * \alpha)$ is an S-contraction with weight st.

The fact that $s\beta$ and $(\beta * \alpha)$ are compatible with the first morphism in (3.1) follows from the following calculation.

$$\begin{pmatrix} \mathbb{1}_{Y_{\ell+1}} \\ 0 \end{pmatrix} s_\nu \beta^\nu_\ell = \begin{pmatrix} s_\nu \beta^\nu_\ell & \beta^\nu_\ell \phi_\ell \alpha^\nu_{\ell-1} \\ 0 & -t_\nu \alpha^\nu_{\ell-1} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{Y_\ell} \\ 0 \end{pmatrix}.$$

Similarly, the fact that $(\beta * \alpha)$ and $\Sigma t\alpha$ are compatible with the second morphism in (3.1) follows from

$$\begin{pmatrix} 0 & \mathbb{1}_{X_\ell} \end{pmatrix} \begin{pmatrix} s_\nu \beta^\nu_\ell & \beta^\nu_\ell \phi_\ell \alpha^\nu_{\ell-1} \\ 0 & -t_\nu \alpha^\nu_{\ell-1} \end{pmatrix} = -t_\nu \alpha^\nu_{\ell-1} \begin{pmatrix} 0 & \mathbb{1}_{X_{\ell-1}} \end{pmatrix}. \qquad \square$$

3.2 Koszul complexes

Throughout this section, d denotes a positive integer and $S = (S_1, \dots, S_d)$ denotes a d-tuple of multiplicative systems of R. Furthermore, e denotes an integer with $e > d$.

The construction of the inverse of ι involves the introduction of a complex $\Delta_e(X, s)$, which would generally be known as the *Koszul complex* of the sequence $s = (s_1, \dots, s_d)$ with coefficients in X_e. More specifically, given a complex $X \in \mathcal{C}_e^R$ and a sequence $s = (s_1, \dots, s_d) \in S_1 \times \cdots \times S_d$, $\Delta_e(X, s)$ is the complex $\Sigma^{e-d} K(s, X_e)$: that is, the Koszul complex of the sequence $s = (s_1, \dots, s_d)$ with coefficients in X_e and shifted $e - d$ degrees to the left. As the reader is not expected to be familiar with Koszul complexes, they are briefly introduced and a few basic results that will be needed later are listed. Furthermore, $\Delta_e(X, s)$ is explicitly described.

Given an element $r \in R$, the *Koszul complex* $K(r)$ of r is the complex

$$0 \longrightarrow R \xrightarrow{r} R \longrightarrow 0$$

concentrated in degrees 1 and 0. Given a sequence $x = (x_1, \dots, x_n) \in R^n$, the *Koszul complex of* x is the complex

$$K(x) = K(x_1, \dots, x_n) = K(x_1) \otimes_R \cdots \otimes_R K(x_n),$$

which is concentrated in degrees $n, \dots, 0$. (The tensor product of complexes is defined in the preliminaries.) Given a sequence $x = (x_1, \dots, x_n) \in R^n$ and a module M, the *Koszul complex of x with coefficients in M* is the complex $K(x, M) = K(x) \otimes_R M$, which is concentrated in degrees $n, \dots, 0$. Note that $K(x, R) = K(x)$.

One of the most important properties of Koszul complexes is that they serve as free resolutions for regular sequences (cf. [BH93, Corollary 1.6.14 and Proposition 1.6.5]).

Theorem 3.6. *Suppose that $x = (x_1, \dots, x_n)$ is a sequence of elements from R, and that M is a module.*

(i) *The homology of $K(x, M)$ is annihilated by the ideal $\langle x \rangle$.*

(ii) *If x is an M-sequence, then the homology of $K(x, M)$ is concentrated in degree 0.*

(iii) *If x is a regular sequence, then $K(x)$ is a finite free resolution of $R/\langle x \rangle$.* □

Since $\operatorname{grade}_R R/\langle x \rangle \leq \operatorname{pd}_R R/\langle x \rangle$, it follows from part (iii) of Theorem 3.6 that $\operatorname{pd}_R R/\langle x \rangle = n$ whenever $x = (x_1, \dots, x_n)$ is a regular sequence.

Given a complex $X \in \mathcal{C}_e^R$ and a sequence $s = (s_1, \dots, s_d) \in S_1 \times \cdots \times S_d$, we define, as mentioned above, $\Delta_e(X, s)$ to be the complex $\Sigma^{e-d} K(s, X_e)$. For convenience we will now present a more explicit description of $\Delta_e(X, s)$.

For any $\ell \in \mathbb{Z}$, let $\Upsilon(\ell)$ denote the set of ℓ-element subsets of $\{1, \dots, d\}$: that is, the set of subsets in the form $i = \{i_1, \dots, i_\ell\}$ where $1 \leq i_1 < \cdots < i_\ell \leq d$. In particular, $\Upsilon(0) = \{\emptyset\}$, $\Upsilon(d) = \{\{1, \dots, d\}\}$ and $\Upsilon(\ell) = \emptyset$ for all $\ell \notin \{0, \dots, d\}$. Thus, in any case, $\Upsilon(\ell)$ contains $\binom{d}{\ell}$ elements. An element $i \in \Upsilon(\ell)$ is called a *multi-index*. The elements of such a multi-index are always denoted by $i_1, \dots, i_\ell$ in increasing order, so that $i = \{i_1, \dots, i_\ell\}$, where $1 \leq i_1 < \cdots < i_\ell \leq d$.

Given a complex $X \in \mathcal{C}_e^R$ and a d-tuple $s = (s_1, \dots, s_d) \in S_1 \times \cdots \times S_d$, $\Delta_e(X, s)$ is the complex whose ℓ'th module is given by

$$\Delta_e(X, s)_\ell = \coprod_{i \in \Upsilon(e-\ell)} \Delta_e(X, s)_\ell^i, \quad \text{where} \quad \Delta_e(X, s)_\ell^i = X_e,$$

and whose ℓ'th differential $\partial_\ell^{\Delta_e(X,s)} \colon \Delta_e(X, s)_\ell \to \Delta_e(X, s)_{\ell-1}$ is given by the fact that its (j, i)-entry $(\partial_\ell^{\Delta_e(X,s)})_{j,i}$ for $i \in \Upsilon(e - \ell)$ and $j \in \Upsilon(e - \ell + 1)$ is

$$(\partial_\ell^{\Delta_e(X,s)})_{j,i} = \begin{cases} (-1)^{u+1} s_{j_u} \mathbb{1}_{X_e}, & \text{if } j \backslash i = \{j_u\} \\ 0, & \text{if } j \not\supseteq i \end{cases}$$

So $\Delta_e(X, s)$ is a complex whose ℓ'th module $\Delta_e(X, s)_\ell$ consists of $\binom{d}{e-\ell}$ copies of X_e and whose ℓ'th differential as a map from the i'th copy of X_e in $\Delta_e(X, s)_\ell$ to the j'th copy of X_e in $\Delta_e(X, s)_{\ell+1}$ is nonzero only when $i \subseteq j$, in which case it is multiplication by $(-1)^{u+1} s_{j_u}$ for the unique j_u which is in j and not in i.

The complex $\Delta_e(X, s)$ comes naturally equipped with an S-contraction.

Theorem 3.7. *If $X \in \mathcal{C}_e^R$ and $s = (s_1, \dots, s_d) \in S_1 \times \cdots \times S_d$, then $\Delta_e(X, s)$ has an S-contraction $\delta_e(X, s)$ with weight s; for each $\ell \in \mathbb{Z}$ and each $\nu = 1, \dots, d$, the homomorphism $\delta_e(X, s)_\ell^\nu \colon \Delta_e(X, s)_\ell \to \Delta_e(X, s)_{\ell+1}$ is given by the fact that its (j, i)-entry for $i \in \Upsilon(e - \ell)$ and $j \in \Upsilon(e - \ell - 1)$ is*

$$(\delta_e(X, s)_\ell^\nu)_{j,i} = \begin{cases} (-1)^{w+1} \mathbb{1}_{X_e}, & \text{if } i \backslash j = \{i_w\} = \{\nu\}, \\ 0, & \text{if } i \not\supseteq j. \end{cases}$$

Proof: This is a matter of verification. For each $\nu \in \{1, \dots, d\}$, $\ell \in \mathbb{Z}$ and $i, i' \in \Upsilon(d - \ell)$, the (i', i)-entry of $\partial_{\ell+1}^{\Delta_e(X,s)} \delta_e(X, s)_\ell^\nu$ is

$$\begin{array}{ll} s_\nu \mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } \nu \in i, \\ (-1)^{u+w} s_{i'_u} \mathbb{1}_{X_e}, & \text{if } i \backslash i' = \{i_w\} = \{\nu\} \text{ and } i' \backslash i = \{i'_u\}, \text{ and} \\ 0, & \text{otherwise,} \end{array}$$

whereas the (i', i)-entry of $\delta_e(X, s)_{\ell-1}^\nu \partial_\ell^{\Delta_e(X,s)}$ is

$$\begin{array}{ll} s_\nu \mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } \nu \notin i, \\ (-1)^{u+w+1} s_{i'_u} \mathbb{1}_{X_e}, & \text{if } i \backslash i' = \{i_w\} = \{\nu\} \text{ and } i' \backslash i = \{i'_u\}, \text{ and} \\ 0, & \text{otherwise.} \end{array}$$

Overall, we see that the (i', i)-entry of $\partial_{\ell+1}^{\Delta_e(X,s)}\delta_e(X,s)^\nu_\ell + \delta_e(X,s)^\nu_{\ell-1}\partial_\ell^{\Delta_e(X,s)}$ is $s_\nu \mathbb{1}_{X_e}$ if $i = i'$ and 0 otherwise. This is what we wanted to show. □

We will only consider $\Delta_e(X, s)$ for complexes X that are homologically S-torsion. It is an important fact that $\Delta_e(X, s)$ is then also homologically S-torsion.

Proposition 3.8. *Let $X \in \mathcal{C}_e^R(\text{f,P}|S\text{-tor})$ and let $s = (s_1, \ldots, s_d) \in S_1 \times \cdots \times S_d$. Then the complex $\Delta_e(X, s)$ is in $\mathcal{C}^R_{[e,e-d]}(\text{f,P}|S\text{-tor})$.*

Proof: The definition clearly implies that $\Delta_e(X, s)$ is concentrated in degrees $e, \ldots, e-d$ and consists of finitely generated projective modules. According to Theorem 3.6(i), the homology modules of $\Delta_e(X, s)$ are annihilated by the ideal $\langle s_1, \ldots, s_d\rangle$. In particular, these homology modules must be S_ν-torsion for $\nu = 1, \ldots, d$. □

3.3 The main theorem

Throughout this section, d denotes a positive integer and $S = (S_1, \ldots, S_d)$ denotes a d-tuple of multiplicative systems of R. Furthermore, X denotes a complex in $\mathcal{C}_e^R(\text{f,P}|S\text{-tor})$ for some integer $e > d$. Finally, α denotes an S-contraction of X with weight $s \in S_1 \times \cdots \times S_d$.

Proving the main theorem requires, within the Grothendieck group $G_e^R(\text{f,P}|S\text{-tor})$, transforming a complex $X \in \mathcal{C}_\square^R(\text{f,P}|S\text{-tor})$ into one or more complexes that are concentrated in fewer degrees than X. We first present the idea behind the proof.

Definition 3.9. Let $\phi_e(X, \alpha)$ denote the family $(\phi_e(X, \alpha)_\ell)_{\ell\in\mathbb{Z}}$ of homomorphisms $\phi_e(X, \alpha)_\ell \colon X_\ell \to \Delta_e(X, s)_\ell = \coprod_{i\in\Upsilon(e-\ell)} X_e$ given by the fact that their i'th entries for $i \in \Upsilon(e-\ell)$ are

$$\phi_e(X, \alpha)^i_\ell = \alpha^{i_{e-\ell}}_{e-1}\alpha^{i_{e-\ell-1}}_{e-2}\cdots\alpha^{i_1}_\ell.$$

For $\ell = e$, this means that $\phi_e(X, \alpha)_e = \mathbb{1}_{X_e}$, and for $\ell \notin \{e, \ldots, e-d\}$, it means that $\phi_e(X, \alpha)_\ell = 0$.

Proposition 3.10. *$\phi_e(X, \alpha) \colon X \to \Delta_e(X, s)$ is a morphism of complexes.*

Proof: Let $\Delta \overset{\text{def}}{=} \Delta_e(X, s)$ and $\phi \overset{\text{def}}{=} \phi_e(X, \alpha)$. To prove that ϕ is a morphism, we need to show that $\phi_{\ell-1}\partial^X_\ell = \partial^\Delta_\ell\phi_\ell$ for all $\ell \in \mathbb{Z}$: that is, we need to verify that the j'th entry, $\alpha^{j_{e-\ell+1}}_{e-1}\cdots\alpha^{j_1}_{\ell-1}\partial^X_\ell$, of the left side equals the j'th entry of $\partial^\Delta_\ell\phi_\ell$ for each $j \in \Upsilon(e-\ell+1)$. Since the (j, i)-entry of ∂^Δ_ℓ is $(-1)^{u+1}s_{j_u}\mathbb{1}_{X_e}$ whenever i is a subset of j with $j\backslash i = \{j_u\}$, that is, whenever $i = \{j_1, \ldots, j_{u-1}, j_{u+1}, \ldots, j_{e-\ell+1}\}$ for some $u \in \{1, \ldots, e-\ell+1\}$, we see that the j'th coordinate of $\partial^\Delta_\ell\phi_\ell$ must be

$$\sum_{u=1}^{e-\ell+1}(-1)^{u+1}s_{j_u}\alpha^{j_{e-\ell+1}}_{e-1}\cdots\alpha^{j_{u+1}}_{\ell+u-1}\alpha^{j_{u-1}}_{\ell+u-2}\cdots\alpha^{j_1}_\ell.$$

So overall, we need to show that

$$\sum_{u=1}^{e-\ell+1} (-1)^{u+1} s_{j_u} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell+u-1}^{j_{u+1}} \alpha_{\ell+u-2}^{j_{u-1}} \cdots \alpha_{\ell}^{j_1} = \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell-1}^{j_1} \partial_{\ell}^{X} \tag{3.2}$$

for all $j \in \Upsilon(e-\ell+1)$. We do this by descending induction on ℓ.

When $\ell > e$, the equation clearly holds since both sides are trivial, and in the case that $\ell = e$, (3.2) states that $s_{j_1} \mathbb{1}_{X_e} = \alpha_{e-1}^{j_1} \partial_e^X$, which is satisfied since α is an S-contraction of X with weight s. Suppose now that $\ell < e$ is arbitrarily chosen and that (3.2) holds for larger values of ℓ. We then have

$$\begin{aligned}
\alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell-1}^{j_1} \partial_{\ell}^{X} &= \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell}^{j_2} (s_{j_1} \mathbb{1}_{X_\ell} - \partial_{\ell+1}^{X} \alpha_{\ell}^{j_1}) \\
&= s_{j_1} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell}^{j_2} \\
&\quad - \Big(\sum_{u=2}^{e-\ell+1} (-1)^{u} s_{j_u} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell+u-1}^{j_{u+1}} \alpha_{\ell+u-2}^{j_{u-1}} \cdots \alpha_{\ell+1}^{j_2} \Big) \alpha_{\ell}^{j_1} \\
&= \sum_{u=1}^{e-\ell+1} (-1)^{u+1} s_{j_u} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell+u-1}^{j_{u+1}} \alpha_{\ell+u-2}^{j_{u-1}} \cdots \alpha_{\ell}^{j_1}.
\end{aligned}$$

Here the second equality follows from the induction hypothesis. This proves (3.2) by induction, so ϕ is a morphism of complexes. □

Definition 3.11. The mapping cone $\mathcal{M}(\phi_e(X,\alpha))$ of $\phi_e(X,\alpha)$ is denoted by $\mathcal{M}_e(X,\alpha)$.

Letting $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$ and $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$, we see that $\mathcal{M}_e(X,\alpha)$ is the complex

$$0 \longrightarrow X_e \xrightarrow{\begin{pmatrix} \phi_e \\ -\partial_e^X \end{pmatrix}} \begin{matrix} \Delta_e \\ \oplus \\ X_{e-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_e^\Delta & \phi_{e-1} \\ 0 & -\partial_{e-1}^X \end{pmatrix}} \begin{matrix} \Delta_{e-1} \\ \oplus \\ X_{e-2} \end{matrix} \longrightarrow \cdots$$

$$\cdots \longrightarrow \begin{matrix} \Delta_{e-d} \\ \oplus \\ X_{e-d-1} \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -\partial_{e-d-1}^X \end{pmatrix}} X_{e-d-2} \xrightarrow{-\partial_{e-d-2}^X} \cdots \longrightarrow X_0 \longrightarrow 0$$

concentrated in degrees $e+1, \ldots, 1$.

Since X and $\Delta_e(X,s)$ are equipped with S-contractions, Theorem 3.5 provides an S-contraction of $\mathcal{M}_e(X,\alpha)$.

Definition 3.12. The S-contraction $\delta_e(X,s) * \alpha$ of $\mathcal{M}_e(X,\alpha)$ is denoted by $\mu_e(X,\alpha)$.

Letting $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$, $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$ and $\delta \overset{\text{def}}{=} \delta_e(X,s)$, we see that $\mu_e(X,\alpha)$ is given by

$$\mu_e(X,\alpha)^\nu_\ell = \begin{pmatrix} s_\nu \delta^\nu_\ell & \delta^\nu_\ell \phi_\ell \alpha^\nu_{\ell-1} \\ & \\ 0 & -s_\nu \alpha^\nu_{\ell-1} \end{pmatrix} : \begin{matrix} \Delta_\ell \\ \oplus \\ X_{\ell-1} \end{matrix} \to \begin{matrix} \Delta_{\ell+1} \\ \oplus \\ X_\ell \end{matrix}$$

for each $\ell \in \mathbb{Z}$ and $\nu \in \{1,\dots,d\}$. The weight of $\mu_e(X,\alpha)$ is $s^2 = (s_1^2,\dots,s_d^2)$.

Proposition 3.13. *$\mathcal{M}_e(X,\alpha)$ is an object of $\mathcal{C}^R_{[e+1,1]}$(f,P|S-tor).*

Proof: $\mathcal{M}_e(X,\alpha)$ is clearly concentrated in degrees $e,\dots,1$ and composed of finitely generated projective modules. To see that $\mathcal{M}_e(X,\alpha)$ is homologically S-torsion, recall that the short exact sequence

$$0 \to \Delta_e(X,s) \to \mathcal{M}_e(X,\alpha) \to \Sigma X \to 0 \tag{3.3}$$

from Theorem 0.2 induces the long exact sequence

$$\cdots \to \mathrm{H}_\ell(\Delta_e(X,s)) \to \mathrm{H}_\ell(\mathcal{M}_e(X,\alpha)) \to \mathrm{H}_\ell(\Sigma X) \to \cdots$$

on homology. Localization preserves exactness, so by localizing at S_ν for $\nu = 1,\dots,d$ it follows that, since $\Delta_e(X,s)$ as well as ΣX are homologically S-torsion, $\mathcal{M}_e(X,\alpha)$ must be homologically S-torsion as well. □

The exact sequence in (3.3) can be shifted one degree to the right, thereby yielding the exact sequence

$$0 \to \Sigma^{-1}\Delta_e(X,s) \to \Sigma^{-1}\mathcal{M}_e(X,\alpha) \to X \to 0$$

in $\mathcal{C}^R_e$(f,P|S-tor), from which it follows that

$$[X] = [\Sigma^{-1}\mathcal{M}_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)]$$

holds in G^R_e(f,P|S-tor). We are trying to represent X by smaller complexes, but $\Sigma^{-1}\mathcal{M}_e(X,\alpha)$ is not in any way smaller than X. However, as shown below, $\Sigma^{-1}\mathcal{M}_e(X,\alpha)$ can be transformed into something that is smaller.

Definition 3.14. Let $\partial^{\mathcal{N}}_{e-1}$ denote the homomorphism

$$\partial^{\mathcal{N}}_{e-1} = \begin{pmatrix} -\phi_e(X,\alpha)_{e-1} \\ \\ \partial^X_{e-1} \end{pmatrix} : X_{e-1} \longrightarrow \begin{matrix} \Delta_e(X,s)_{e-1} \\ \oplus \\ X_{e-2} \end{matrix} = \mathcal{M}_e(X,\alpha)_{e-1},$$

and let $\mathcal{N}_e(X,\alpha)$ denote the complex

$$0 \longrightarrow X_{e-1} \xrightarrow{\partial^{\mathcal{N}}_{e-1}} \mathcal{M}_e(X,\alpha)_{e-1} \xrightarrow{-\partial^{\mathcal{M}_e(X,\alpha)}_{e-1}} \mathcal{M}_e(X,\alpha)_{e-2} \xrightarrow{-\partial^{\mathcal{M}_e(X,\alpha)}_{e-2}} \cdots \longrightarrow \mathcal{M}_e(X,\alpha)_1 \longrightarrow 0$$

concentrated in degrees $e-1,\dots,0$.

(One verifies easily that $\mathcal{N}_e(X,\alpha)$ indeed is a complex.)

Proposition 3.15. $\mathcal{N}_e(X,\alpha)$ *is an object of* $\mathcal{C}^R_{e-1}$(f,P|S-tor).

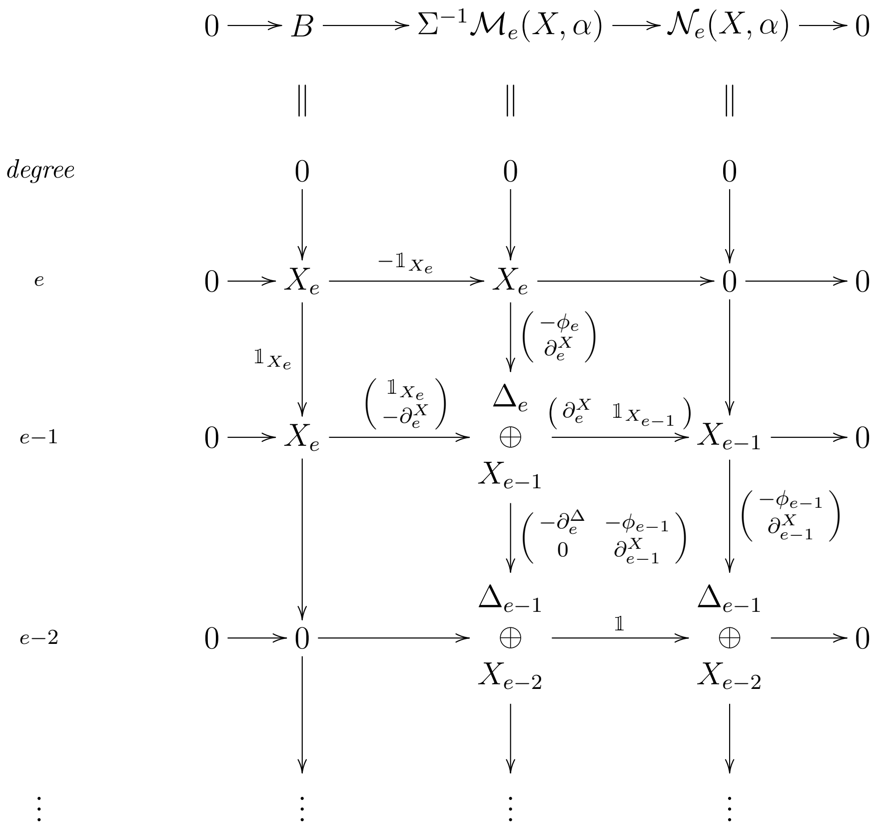
PROOF: $\mathcal{N}_e(X,\alpha)$ is clearly composed of finitely generated projective modules. The fact that $\mathcal{N}_e(X,\alpha)$ is homologically S-torsion follows from Proposition 3.13 and Theorem 3.16 below, from which it follows that $\mathcal{N}_e(X,\alpha)$ is homologically isomorphic to $\Sigma^{-1}\mathcal{M}_e(X,\alpha)$. $\square$

$\mathcal{N}_e(X,\alpha)$ is $\mathcal{M}_e(X,\alpha)$ without the segment $0 \to X_e \to X_e \to 0$ and shifted one degree to the right. The good thing about $\mathcal{N}_e(X,\alpha)$ is that it is smaller than $\mathcal{M}_e(X,\alpha)$, so we hope that we in some way at the level of Grothendieck groups can represent $\mathcal{M}_e(X,\alpha)$ by $\mathcal{N}_e(X,\alpha)$. This is achieved in the theorem below.

Theorem 3.16. *Let B denote the exact complex $0 \to X_e \to X_e \to 0$ concentrated in degrees e and $e-1$. There is then an exact sequence*

$$0 \to B \to \Sigma^{-1}\mathcal{M}_e(X,\alpha) \to \mathcal{N}_e(X,\alpha) \to 0.$$

PROOF: Let $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$ and $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$, and recall that $\Delta_e = X_e$ and $\phi_e = \mathbb{1}_{X_e}$. The situation is as follows.



It is straightforward to verify that the diagram commutes and that all the rows are exact. □

The morphism $\Sigma^{-1}\mathcal{M}_e(X,\alpha) \to \mathcal{N}_e(X,\alpha)$ from Theorem 3.16 is clearly in the form described in Theorem 3.4, so we are able to induce an S-contraction on $\mathcal{N}_e(X,\alpha)$ with weight s^2 from the S-contraction $\Sigma^{-1}\mu_e(X,\alpha)$ on $\Sigma^{-1}\mathcal{M}_e(X,\alpha)$.

Definition 3.17. The S-contraction on $\mathcal{N}_e(X,\alpha)$ induced in the sense of Theorem 3.4 from $\Sigma^{-1}\mu_e(X,\alpha)$ through the morphism $\Sigma^{-1}\mathcal{M}_e(X,\alpha) \to \mathcal{N}_e(X,\alpha)$ from Theorem 3.16 is denoted by $\eta_e(X,\alpha)$.

Letting $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$, $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$ and $\delta \overset{\text{def}}{=} \delta_e(X,s)$, $\eta_e(X,\alpha)$ from the above definition is given by

$$\eta_e(X,\alpha)^\nu_\ell = \begin{pmatrix} -s_\nu\delta^\nu_{\ell+1} & -\delta_{\ell+1}\phi_{\ell+1}\alpha^\nu_\ell \\ 0 & s_\nu\alpha^\nu_\ell \end{pmatrix} : \begin{matrix} \Delta_e(X,s)_{\ell+1} \\ \oplus \\ X_\ell \end{matrix} \to \begin{matrix} \Delta_e(X,s)_{\ell+2} \\ \oplus \\ X_{\ell+1} \end{matrix}$$

whenever $\ell = e-3,\dots,0$, and, as verified by a small calculation, by

$$\eta_e(X,\alpha)^\nu_{e-2} = \begin{pmatrix} -s_\nu\partial^X_e\delta^\nu_{e-1} & \alpha^\nu_{e-2}\partial^X_{e-1}\alpha^\nu_{e-2} \end{pmatrix} : \begin{matrix} \Delta_{e-1} \\ \oplus \\ X_{e-2} \end{matrix} \to X_{e-1}$$

whenever $\ell = e-2$.

From Theorem 3.16 it follows that

$$[\Sigma^{-1}\mathcal{M}_e(X,\alpha)] = [B] + [\mathcal{N}_e(X,\alpha)] = [\mathcal{N}_e(X,\alpha)]$$

in $G^R_e(\text{f,P}|S\text{-tor})$, so together with the previous work, we have now succeeded in transforming our complex X into something smaller at the level of Grothendieck groups:

$$[X] = [\Sigma^{-1}\mathcal{M}_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)] = [\mathcal{N}_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)].$$

Although this is an equation in $G^R_e(\text{f,P}|S\text{-tor})$, the complexes involved in the right end are both concentrated in fewer than e degrees. This, at least, gives us an idea of how to construct the inverse of ι.

Definition 3.18. By $w_e(X,\alpha)$ we denote the element

$$w_e(X,\alpha) = [\mathcal{N}_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)]$$

in $G^R_{e-1}(\text{f,P}|S\text{-tor})$.

The remainder of this section is devoted to showing that $w_e(X,\alpha)$ is independent of the choice of α such that we can simply write $w_e(X)$; that the map $w_e\colon \mathcal{I}(\mathcal{C}_e^R(\text{f,P}|S\text{-tor})) \to G_{e-1}^R(\text{f,P}|S\text{-tor})$ induces a map $\omega_e\colon G_e^R(\text{f,P}|S\text{-tor}) \to G_{e-1}^R(\text{f,P}|S\text{-tor})$; and that the ω_e's for different e's can be combined to form an inverse of ι.

We begin with a collection of useful lemmas.

Lemma 3.19. *If*

$$0 \longrightarrow \bar{Y} \xrightarrow{\bar{\psi}} Y \xrightarrow{\psi} \tilde{Y} \longrightarrow 0$$

is an exact sequence in $\mathcal{C}_e^R(\text{f,P}|S\text{-tor})$*, and if* $\bar{\beta}$*,* β *and* $\tilde{\beta}$ *are* S*-contractions of* $\bar{Y}$*,* Y *and* $\tilde{Y}$*, respectively, compatible with the morphisms in the above exact sequence (and thereby all having the same weight* t*), then there are exact sequences*

$$0 \to \Delta_e(\bar{Y},t) \to \Delta_e(Y,t) \to \Delta_e(\tilde{Y},t) \to 0, \tag{3.4}$$

$$0 \to \mathcal{M}_e(\bar{Y},\bar{\beta}) \to \mathcal{M}_e(Y,\beta) \to \mathcal{M}_e(\tilde{Y},\tilde{\beta}) \to 0 \text{ and} \tag{3.5}$$

$$0 \to \mathcal{N}_e(\bar{Y},\bar{\beta}) \to \mathcal{N}_e(Y,\beta) \to \mathcal{N}_e(\tilde{Y},\tilde{\beta}) \to 0, \tag{3.6}$$

proving that $w_e(Y,\beta) = w_e(\bar{Y},\bar{\beta}) + w_e(\tilde{Y},\tilde{\beta})$ *in* $G_{e-1}^R(\text{f,P}|S\text{-tor})$*. Furthermore the* S*-contractions* $\delta_e(\bar{Y},t)$*,* $\delta_e(Y,t)$ *and* $\delta_e(\tilde{Y},t)$ *are compatible with the morphisms in* (3.4)*; the* S*-contractions* $\mu_e(\bar{Y},\bar{\beta})$*,* $\mu_e(Y,\beta)$ *and* $\mu_e(\tilde{Y},\tilde{\beta})$ *are compatible with the morphisms in* (3.5)*; and the* S*-contractions* $\eta_e(\bar{Y},\bar{\beta})$*,* $\eta_e(Y,\beta)$ *and* $\eta_e(\tilde{Y},\tilde{\beta})$ *are compatible with the morphisms in* (3.6)*.*

Proof: According to the assumption, there is an exact sequence of modules

$$0 \longrightarrow \bar{Y}_e \xrightarrow{\bar{\psi}_e} Y_e \xrightarrow{\psi_e} \tilde{Y}_e \longrightarrow 0,$$

which immediately induces the exact sequence in (3.4), because $\bar{\psi}_e$ and ψ_e clearly commute with each entry of the differentials in $\Delta_e(\bar{Y},t)$, $\Delta_e(Y,t)$ and $\Delta_e(\tilde{Y},t)$. Since $\bar{\psi}_e$ and ψ_e also commute with each entry of the the S-contractions $\delta_e(\bar{Y},t)$, $\delta_e(Y,t)$ and $\delta_e(\tilde{Y},t)$, these must be compatible with the morphisms in the sequence. In addition, the compatibility of the S-contractions $\bar{\beta}$, β and $\tilde{\beta}$ with the morphisms $\bar{\psi}$ and ψ means that $\bar{\psi}_e\phi_e(\bar{Y},\bar{\beta})^i_\ell = \phi_e(Y,\beta)^i_\ell\bar{\psi}_\ell$ and $\psi_e\phi_e(Y,\beta)^i_\ell = \phi_e(\tilde{Y},\tilde{\beta})^i_\ell\psi_\ell$ for each $\ell \in \mathbb{Z}$ and $i \in \Upsilon(e-\ell)$, and hence that there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{Y} & \longrightarrow & Y & \longrightarrow & \tilde{Y} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_e(\bar{Y},\bar{\beta})} & & \downarrow{\scriptstyle \phi_e(Y,\beta)} & & \downarrow{\scriptstyle \phi_e(\tilde{Y},\tilde{\beta})} & & \\
0 & \longrightarrow & \Delta_e(\bar{Y},s) & \longrightarrow & \Delta_e(Y,s) & \longrightarrow & \Delta_e(\tilde{Y},s) & \longrightarrow & 0
\end{array} \tag{3.7}$$

From Theorem 0.3 now follows the existence of the exact sequence in (3.5), and straightforward calculation easily verifies that the compatibility of the S-contractions $\bar{\beta}$, β and $\tilde{\beta}$ with the morphisms $\bar{\psi}$ and ψ, the compatibility of the

S-contractions $\delta_e(\bar{Y},t)$, $\delta_e(Y,t)$ and $\delta_e(\tilde{Y},t)$ with the morphisms in (3.4) and the commutativity of diagram (3.7) imply that the S-contractions $\mu_e(\bar{Y},\bar{\beta})$, $\mu_e(Y,\beta)$ and $\mu_e(\tilde{Y},\tilde{\beta})$ are compatible with the morphisms in (3.5).

We now claim that the exact sequence in (3.5) induces the exact sequence in (3.6). To see this, let $\bar{B}$, B and $\tilde{B}$ denote the complexes $0 \to \bar{Y}_e \to \bar{Y}_e \to 0$, $0 \to Y_e \to Y_e \to 0$ and $0 \to \tilde{Y}_e \to \tilde{Y}_e \to 0$ concentrated in degrees e and $e-1$ from Theorem 3.16. These three complexes come together in a short exact sequence $0 \to \bar{B} \to B \to \tilde{B} \to 0$, induced by the short exact sequence $0 \to \bar{Y}_e \to Y_e \to \tilde{Y}_e \to 0$. We claim that there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{B} & \longrightarrow & B & \longrightarrow & \tilde{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{-1}\mathcal{M}_e(\bar{Y},\bar{\beta}) & \longrightarrow & \Sigma^{-1}\mathcal{M}_e(Y,\beta) & \longrightarrow & \Sigma^{-1}\mathcal{M}_e(\tilde{Y},\tilde{\beta}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_e(\bar{Y},\bar{\beta}) & \longrightarrow & \mathcal{N}_e(Y,\beta) & \longrightarrow & \mathcal{N}_e(\tilde{Y},\tilde{\beta}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The columns are exact according to Theorem 3.16 and the top rectangles are readily verified to be commutative. A little diagram chase now shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, making the entire diagram commutative by construction. As we have seen, the two top rows are exact, so the exactness of the bottom row follows from the 9-lemma (see, for example, [Eis95, exercise A3.12]) applied in each degree. This establishes the exact sequence in (3.6). Once again, straightforward calculation demonstrates that the S-contractions $\eta_e(\bar{Y},\bar{\beta})$, $\eta_e(Y,\beta)$ and $\eta_e(\tilde{Y},\tilde{\beta})$ are compatible with the morphisms in (3.6).

From (3.4) and (3.6), we now obtain that

$$\begin{aligned}
w_e(Y,\beta) &= [\mathcal{N}_e(Y,\beta)] - [\Sigma^{-1}\Delta_e(Y,t)] \\
&= [\mathcal{N}_e(\bar{Y},\bar{\beta})] + [\mathcal{N}_e(\tilde{Y},\tilde{\beta})] - [\Sigma^{-1}\Delta_e(\bar{Y},t)] - [\Sigma^{-1}\Delta_e(\tilde{Y},t)] \\
&= w_e(\bar{Y},\bar{\beta}) + w_e(\tilde{Y},\tilde{\beta}),
\end{aligned}$$

and the proof is complete. □

Lemma 3.20. *If X is exact, then $w_e(X,\alpha) = 0$ in G^R_{e-1}(f,P|S-tor).*

Proof: Let $\tilde{\partial}^X_{e-1}$ denote the inclusion map $\operatorname{im}\partial^X_{e-1} \hookrightarrow X_{e-2}$, and let $\tilde{X}$ denote the complex

$$0 \longrightarrow \operatorname{im}\partial^X_{e-1} \xrightarrow{\tilde{\partial}^X_{e-1}} X_{e-2} \xrightarrow{\partial^X_{e-2}} X_{e-3} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\partial^X_1} X_0 \longrightarrow 0$$

concentrated in degrees $e-1,\dots 0$. $\tilde{X}$ is X with the module in degree e removed and the module in degree $e-1$ replaced by $\operatorname{im}\partial^X_{e-1}$. Since X is exact, $\tilde{X}$ is exact and it follows from Theorem 0.5 that $\operatorname{im}\partial^X_{e-1}$ is projective, and hence that $\tilde{X}$ is a complex in $\mathcal{C}^R_{e-1}(\text{f,P}|S\text{-tor})$.

Letting B denote the exact complex $0 \to X_e \to X_e \to 0$ from Theorem 3.16, there is an exact sequence

$$\begin{array}{lccccccccc}
 & 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & \tilde{X} & \longrightarrow & 0 \\
 & & & \| & & \| & & \| & & \\
\textit{degree} & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
e & 0 & \longrightarrow & X_e & \xrightarrow{\mathbb{1}_{X_e}} & X_e & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \mathbb{1}_{X_e}} & & \downarrow{\scriptstyle \partial^X_e} & & \downarrow & & \\
e-1 & 0 & \longrightarrow & X_e & \xrightarrow{\partial^X_{e-1}} & X_{e-1} & \xrightarrow{\partial^X_{e-1}} & \operatorname{im}\partial^X_{e-1} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow{\scriptstyle \partial^X_{e-1}} & & \downarrow{\scriptstyle \tilde{\partial}^X_{e-1}} & & \\
e-2 & 0 & \longrightarrow & 0 & \longrightarrow & X_{e-2} & \xrightarrow{\mathbb{1}_{X_{e-2}}} & X_{e-2} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & & \vdots & & \vdots & & \vdots & &
\end{array}$$

and we claim that there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{-1}\Delta_e(X,s) & \longrightarrow & \Sigma^{-1}\mathcal{M}_e(X,\alpha) & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{-1}\Delta_e(X,s) & \longrightarrow & \mathcal{N}_e(X,\alpha) & \longrightarrow & \tilde{X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The columns are exact (the middle one according to Theorem 3.16) and the top rectangles are readily verified to be commutative. A little diagram chase shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, so that the entire diagram is commutative by construction. Now, the two top rows are exact, so the exactness of the bottom row follows from the 9-lemma (see, for example, [Eis95, exercise A3.12]) applied in each degree. Thus, we have constructed an exact sequence

$$0 \to \Sigma^{-1}\Delta_e(X,s) \to \mathcal{N}_e(X,\alpha) \to \tilde{X} \to 0 \tag{3.8}$$

of complexes in $\mathcal{C}^R_{e-1}$(f,P|S-tor). Since $\tilde{X}$ is exact, it follows that

$$w_e(X,\alpha) = [\mathcal{N}_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)] = [\tilde{X}] = 0$$

in G^R_{e-1}(f,P|S-tor) as desired. □

In the next lemma and the theorem that follows, we shall work with a number of similar Koszul complexes. Let us therefore introduce some convenient notation.

Definition 3.21. For $r \in S_1$, let $\Delta(r) \overset{\text{def}}{=} \Delta_e(X,(r,s_2,\dots,s_d))$; hence, in particular, $\Delta(s_1) = \Delta_e(X,s)$.

Lemma 3.22. *Suppose $r, r' \in S_1$, and define homomorphisms*

$$\pi(r,r')_\ell \colon \Delta(rr')_\ell \to \Delta(r)_\ell \quad \text{and} \quad \xi(r,r') \colon \Delta(r)_\ell \to \Delta(rr')_\ell$$

for each $\ell \in \mathbb{Z}$ by the fact that their (i',i)-entries for $i, i' \in \Upsilon(e-\ell)$ are

$$(\pi(r,r')_\ell)_{i',i} = \begin{cases} 0, & \text{if } i \neq i', \\ \mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \\ r'\mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i, \end{cases}$$

and

$$(\xi(r,r')_\ell)_{i',i} = \begin{cases} 0, & \text{if } i \neq i', \\ r'\mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \\ \mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i. \end{cases}$$

Then $\pi(r,r') = (\pi(r,r')_\ell)_{\ell\in\mathbb{Z}}$ is a morphism of complexes $\Delta(rr') \to \Delta(r)$ and $\xi(r,r') = (\xi(r,r')_\ell)_{\ell\in\mathbb{Z}}$ is a morphism of complexes $\Delta(r) \to \Delta(rr')$.

Proof: Assume that $i \in \Upsilon(e-\ell)$ and $j \in \Upsilon(e-\ell+1)$. A direct calculation then shows that the (j,i)-entries of $\partial^{\Delta(r)}_\ell \pi(r,r')_\ell$ and $\pi(r,r')_{\ell-1}\partial^{\Delta(rr')}_\ell$ are both given by

$$\begin{array}{ll} 0, & \text{if } j \not\supseteq i, \\ (-1)^{u+1}s_{j_u}\mathbb{1}_{X_e}, & \text{if } j\backslash i = \{j_u\} \text{ and } 1 \in i, \\ (-1)^{u+1}s_{j_u}r'\mathbb{1}_{X_e}, & \text{if } j\backslash i = \{j_u\} \neq \{1\} \text{ and } 1 \notin i, \text{ and} \\ rr'\mathbb{1}_{X_e}, & \text{if } j\backslash i = \{j_u\} = \{1\} \text{ and } 1 \notin i. \end{array}$$

This proves that $\pi(r,r')$ is a morphism of complexes.

Similarly, a direct calculation shows that the (j,i)-entries of $\partial_\ell^{\Delta(rr')}\xi(r,r')_\ell$ and $\xi(r,r')_{\ell-1}\partial_\ell^{\Delta(r)}$ are both given by

$$\begin{array}{cl} 0, & \text{if } j \not\supseteq i, \\ (-1)^{u+1}s_{j_u}r'\mathbb{1}_{X_e}, & \text{if } j\backslash i = \{j_u\} \text{ and } 1 \in i, \\ (-1)^{u+1}s_{j_u}\mathbb{1}_{X_e}, & \text{if } j\backslash i = \{j_u\} \neq \{1\} \text{ and } 1 \notin i, \text{ and} \\ rr'\mathbb{1}_{X_e}, & \text{if } j\backslash i = \{j_u\} = \{1\} \text{ and } 1 \notin i. \end{array}$$

This proves that $\xi(r,r')$ is a morphism of complexes. □

We are now ready to take the first step in proving that $w_e(X,\alpha)$ is independent of α.

Theorem 3.23. *Suppose that $t = (t_1,\dots,t_d) \in S_1 \times \cdots \times S_d$ and consider the S-contraction $t\alpha = (t_1\alpha^1,\dots,t_d\alpha^d)$ of X with weight $st = (s_1t_1,\dots,s_dt_d)$. Then $w_e(X,t\alpha) = w_e(X,\alpha)$ in G^R_{e-1}(f,P$|S$-tor).*

Proof: If only we can show the equation in the case where $t_\nu = 1$ for all but one of the ν's, then the equation follows since

$$t\alpha = (t_1,\dots,t_d)\alpha = (t_1,1\dots,1)\cdots(1,\dots,1,t_d)\alpha.$$

We will therefore assume that $t = (t_1,1,\dots,1)$; the other cases follow similarly (since we can permute the S_ν's).

To show the desired equation, it suffices to prove that the following equations hold in G^R_{e-1}(f,P$|S$-tor).

$$[\Sigma^{-1}\Delta(s_1t_1)] = [\Sigma^{-1}\Delta(s_1)] + [\Sigma^{-1}\Delta(t_1)]. \tag{3.9}$$
$$[\mathcal{N}_e(X,t\alpha)] = [\mathcal{N}_e(X,\alpha)] + [\Sigma^{-1}\Delta(t_1)]. \tag{3.10}$$

Since $\Delta(1)$ is exact according to Theorem 3.6(i), the first equation follows if we can show that there is an exact sequence

$$0 \longrightarrow \Delta(s_1) \xrightarrow{\left(\begin{smallmatrix}\pi(1,s_1)\\ \xi(s_1,t_1)\end{smallmatrix}\right)} \begin{matrix}\Delta(1)\\ \oplus \\ \Delta(s_1t_1)\end{matrix} \xrightarrow{(\,-\xi(1,t_1)\ \ \pi(t_1,s_1)\,)} \Delta(t_1) \longrightarrow 0.$$

The two matrices clearly define morphisms of complexes, since $\pi(r,u)$ and $\xi(r,u)$ are morphisms of complexes for $r,u \in S_1$ according to Lemma 3.22. Exactness at $\Delta(s_1)$ and $\Delta(t_1)$ is clear since there is always one identity map involved in either of $\pi(r,u)$ and $\xi(r,u)$ for $r,u \in S_1$. Furthermore, $\xi(1,t_1)\pi(1,s_1)$ as well as $\pi(t_1, s_1)\xi(s_1, t_1)$ are defined in degree ℓ by the fact that their (i, i')-entries for $i, i' \in \Upsilon(e - \ell)$ are

$$\begin{array}{ll} 0, & \text{if } i \neq i', \\ t_1 \mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \text{ and} \\ s_1 \mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i. \end{array}$$

To show the exactness of the sequence above, it therefore only remains to show that, for each $\ell \in \mathbb{Z}$, the kernel in degree ℓ of the second morphism is contained in the image in degree ℓ of the first. Since all (i, i')-entries of the maps involved are trivial except when $i = i'$, it suffices to consider an element (x, y) in the i-entry $\Delta(1)^i_\ell \oplus \Delta(s_1 t_1)^i_\ell$ of the ℓ'th module of $\Delta(1) \oplus \Delta(s_1 t_1)$. So suppose that such an element is in the kernel of the map in degree ℓ of the second morphism. If $1 \in i$, this means that $t_1 x = y$, and in this case (x, y) is the image of x under the map in degree ℓ of the first morphism. If $1 \notin i$, it means that $x = s_1 y$, and in this case (x, y) is the image of y under the map in degree ℓ of the first morphism. In all cases, (x, y) is in the image of the map in degree ℓ of the first morphism, and hence the sequence is exact and equation (3.9) has been proven.

Moving on to equation (3.10), we first define for each $\ell \in \mathbb{Z}$ a homomorphism $\gamma_{\ell-1} \colon X_{\ell-1} \to \Delta(1)_\ell$ by letting its i'th entry for $i \in \Upsilon(e - \ell)$ be

$$\gamma^i_{\ell-1} = \begin{cases} 0, & \text{if } 1 \in i, \\ \alpha^{i_{e-\ell}}_{e-1} \cdots \alpha^{i_1}_\ell \alpha^1_{\ell-1}, & \text{if } 1 \notin i. \end{cases}$$

Another way of writing this is

$$\gamma_{\ell-1} = \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} \alpha^{i_{e-\ell}}_{e-1} \cdots \alpha^{i_1}_\ell \alpha^1_{\ell-1}.$$

We now claim that there are morphisms

$$\Phi \colon \mathcal{M}_e(X, \alpha) \longrightarrow \begin{array}{c} \Delta(1) \\ \oplus \\ \mathcal{M}_e(X, t\alpha) \end{array} \quad \text{and} \quad \Psi \colon \begin{array}{c} \Delta(1) \\ \oplus \\ \mathcal{M}_e(X, t\alpha) \end{array} \longrightarrow \Delta(t_1)$$

given in degree ℓ by

$$\Phi_\ell = \begin{pmatrix} \pi(1, s_1)_\ell & \gamma_{\ell-1} \\ \xi(s_1, t_1)_\ell & 0 \\ 0 & \mathbb{1}_{X_{\ell-1}} \end{pmatrix} : \begin{array}{c} \Delta(s_1)_\ell \\ \oplus \\ X_{\ell-1} \end{array} \longrightarrow \begin{array}{c} \Delta(1)_\ell \\ \oplus \\ \Delta(s_1 t_1)_\ell \\ \oplus \\ X_{\ell-1} \end{array}$$

and

$$\Psi_\ell = \begin{pmatrix} -\xi(1, t_1)_\ell & \pi(t_1, s_1)_\ell & \xi(1, t_1)_\ell \gamma_{\ell-1} \end{pmatrix} : \begin{array}{c} \Delta(1)_\ell \\ \oplus \\ \Delta(s_1 t_1)_\ell \\ \oplus \\ X_{\ell-1} \end{array} \longrightarrow \Delta(t_1)_\ell.$$

Proving that Φ and Ψ indeed are morphisms of complexes means proving that

$$\begin{pmatrix} \partial^{\Delta(1)}_{\ell+1} & 0 & 0 \\ 0 & \partial^{\Delta(s_1t_1)}_{\ell+1} & \phi_e(X,t\alpha)_\ell \\ 0 & 0 & -\partial^X_\ell \end{pmatrix} \Phi_{\ell+1} = \Phi_\ell \begin{pmatrix} \partial^{\Delta(s_1)}_{\ell+1} & \phi_e(X,\alpha)_\ell \\ 0 & -\partial^X_\ell \end{pmatrix}$$

and

$$\partial^{\Delta(t_1)}_{\ell+1}\Psi_{\ell+1} = \Psi_\ell \begin{pmatrix} \partial^{\Delta(1)}_{\ell+1} & 0 & 0 \\ 0 & \partial^{\Delta(s_1t_1)}_{\ell+1} & \phi_e(X,t\alpha)_\ell \\ 0 & 0 & -\partial^X_\ell \end{pmatrix}$$

for all $\ell \in \mathbb{Z}$. Since we already know from Lemma 3.22 that $\pi(r,u)$ and $\xi(r,u)$ are morphisms for $r,u \in S_1$, proving the above equations comes down to showing that the following hold for all $\ell \in \mathbb{Z}$.

$$\pi(1,s_1)_\ell\phi_e(X,\alpha)_\ell = \partial^{\Delta(1)}_{\ell+1}\gamma_\ell + \gamma_{\ell-1}\partial^X_\ell. \tag{3.11}$$

$$\phi_e(X,t\alpha)_\ell = \xi(s_1,t_1)_\ell\phi_e(X,\alpha)_\ell. \tag{3.12}$$

$$\pi(t_1,s_1)_\ell\phi_e(X,t\alpha)_\ell = \xi(1,t_1)_\ell\gamma_{\ell-1}\partial^X_\ell + \partial^{\Delta(t_1)}_{\ell+1}\xi(1,t_1)_{\ell+1}\gamma_\ell. \tag{3.13}$$

We verify (3.11) by brute force, calculating on the right hand side of the equation:

$$\begin{aligned}
\partial^{\Delta(1)}_{\ell+1}\gamma_\ell + \gamma_{\ell-1}\partial^X_\ell &= \partial^{\Delta(1)}_{\ell+1} \coprod_{\substack{j\in\Upsilon(e-\ell-1)\\ 1\notin j}} \alpha^{j_{e-\ell-1}}_{e-1}\cdots\alpha^{j_1}_{\ell+1}\alpha^1_\ell \\
&\quad + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell\alpha^1_{\ell-1}\partial^X_\ell \\
&= \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_2}_{\ell+1}\alpha^1_\ell \\
&\quad + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \Big(\sum_{u=1}^{e-\ell}(-1)^{u+1}s_{i_u}\alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_{u+1}}_{\ell+u}\alpha^{i_{u-1}}_{\ell+u-1}\cdots\alpha^{i_1}_{\ell+1}\alpha^1_\ell\Big) \\
&\quad + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell\alpha^1_{\ell-1}\partial^X_\ell \\
&= \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell \\
&\quad + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell(\partial^X_{\ell+1}\alpha^1_\ell + \alpha^1_{\ell-1}\partial^X_\ell) \\
&= \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} s_1\alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell \\
&= \pi(1,s_1)_\ell\phi_e(X,\alpha)_\ell.
\end{aligned}$$

Here, the third equality follows from (3.2) in Proposition 3.10, and the fourth equality follows from α being an S-contraction with weight $s = (s_1, \dots, s_d)$. This proves the equation in (3.11). The equation in (3.12) is clear, since

$$\xi(s_1,t_1)_\ell \phi_e(X,\alpha)_\ell = \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} t_1\phi_e(X,\alpha)^i_\ell + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \phi_e(X,\alpha)^i_\ell = \phi_e(X,t\alpha).$$

To prove that the equation in (3.13) holds, we apply (3.11) to the right side of (3.13):

$$\begin{aligned}\xi(1,t_1)_\ell\gamma_{\ell-1}\partial^X_\ell &+ \partial^{\Delta(t_1)}_{\ell+1}\xi(1,t_1)_{\ell+1}\gamma_\ell \\ &= \xi(1,t_1)_\ell(\gamma_{\ell-1}\partial^X_\ell + \partial^{\Delta(1)}_{\ell+1}\gamma_\ell) \\ &= \xi(1,t_1)_\ell\pi(1,s_1)_\ell\phi_e(X,\alpha)_\ell.\end{aligned}$$

In contrast, applying (3.12) to the left side of (3.13) yields

$$\pi(t_1,s_1)_\ell\phi_e(X,t\alpha)_\ell = \pi(t_1,s_1)_\ell\xi(s_1,t_1)_\ell\phi_e(X,\alpha)_\ell,$$

so proving equation (3.13) merely requires showing that

$$\xi(1,t_1)_\ell\pi(1,s_1)_\ell = \pi(t_1,s_1)_\ell\xi(s_1,t_1)_\ell. \tag{3.14}$$

This, however, follows since, for $i, i' \in \Upsilon(e-\ell)$, both sides of (3.14) have (i,i')-entries given by

$$\begin{cases} 0, & \text{if } i \neq i', \\ s_1\mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i, \text{ and} \\ t_1\mathbb{1}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i. \end{cases}$$

Thus we have verified equation (3.13), and we conclude that Φ and Ψ are morphisms of complexes.

We now claim that there is a short exact sequence

$$0 \longrightarrow \mathcal{M}_e(X,\alpha) \xrightarrow{\Phi} \begin{matrix}\Delta(1)\\ \oplus \\ \mathcal{M}_e(X,t\alpha)\end{matrix} \xrightarrow{\Psi} \Delta(t_1) \longrightarrow 0. \tag{3.15}$$

To see that the sequence is exact at $\mathcal{M}_e(X,\alpha)$, suppose that, for some $\ell \in \mathbb{Z}$, the element $(x,y) \in \Delta(s_1)_\ell \oplus X_{\ell-1} = \mathcal{M}_e(X,\alpha)_\ell$ maps to 0 under Φ_ℓ: that is,

$$0 = \begin{pmatrix} \pi(1,s_1)_\ell & \gamma_{\ell-1} \\ \xi(s_1,t_1)_\ell & 0 \\ 0 & \mathbb{1}_{X_{\ell-1}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \pi(1,s_1)_\ell(x) + \gamma_{\ell-1}(y) \\ \xi(s_1,t_1)_\ell(x) \\ y \end{pmatrix}.$$

It immediately follows that $y = 0$, and we are left with the equations $\pi(1,s_1)_\ell(x) = \xi(s_1,t_1)_\ell(x) = 0$ which imply that $x = 0$. Thus, Φ_ℓ is injective and (3.15) is exact at $\mathcal{M}_e(X,\alpha)$.

To see that the sequence is exact at $\Delta(t_1)$, suppose that $x \in \Delta(t_1)^i_\ell$ for some $\ell \in \mathbb{Z}$ and $i \in \Upsilon(e-\ell)$. Then, if $1 \in i$,

$$\begin{pmatrix} -\xi(1,t_1)_\ell & \pi(t_1,s_1)_\ell & \xi(1,t_1)_\ell\gamma_{\ell-1} \end{pmatrix} \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} = x,$$

and if $1 \notin i$,

$$\begin{pmatrix} -\xi(1,t_1)_\ell & \pi(t_1,s_1)_\ell & \xi(1,t_1)_\ell\gamma_{\ell-1} \end{pmatrix} \begin{pmatrix} -x \\ 0 \\ 0 \end{pmatrix} = x.$$

In either case, x is in the image of Ψ_ℓ, and we conclude that Ψ_ℓ is surjective and that (3.15) is exact at $\Delta(t_1)$.

Equation (3.14) clearly shows that $\Psi\Phi = 0$, so to show the exactness of (3.15), it only remains verify that the kernel of Ψ_ℓ is contained in the image of Φ_ℓ for all $\ell \in \mathbb{Z}$. So suppose that $(x,y,z) \in \Delta(1)_\ell \oplus \Delta(s_1t_1)_\ell \oplus X_{\ell-1} = (\Delta(1) \oplus \mathcal{M}_e(X,t\alpha))_\ell$ maps to 0 under Ψ_ℓ: that is,

$$-\xi(1,t_1)_\ell(x) + \pi(t_1,s_1)_\ell(y) + \xi(1,t_1)_\ell\gamma_{\ell-1}(z) = 0.$$

Here $x = (x_i)_{i\in\Upsilon(e-\ell)}$ and $y = (y_i)_{i\in\Upsilon(e-\ell)}$ are $\Upsilon(e-\ell)$-tuples, so the above equation states that, for $i \in \Upsilon(e-\ell)$,

$$\begin{aligned} -t_1x_i + y_i &= 0, \quad \text{if } 1 \in i, \text{ and} \\ -x_i + s_1y_i + \gamma^i_{\ell-1}(z) &= 0, \quad \text{if } 1 \notin i. \end{aligned}$$

Now let $w = (w_i)_{i\in\Upsilon(e-\ell)} \in \Delta(s_1)_\ell$ be defined by $w_i = x_i$ whenever $1 \in i$ and $w_i = y_i$ whenever $1 \notin i$. Then

$$\begin{aligned} \begin{pmatrix} \pi(1,s_1)_\ell & \gamma_{\ell-1} \\ \xi(s_1,t_1)_\ell & 0 \\ 0 & \mathbb{1}_{X_{\ell-1}} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} &= \begin{pmatrix} \pi(1,s_1)_\ell(w) + \gamma_{\ell-1}(z) \\ \xi(s_1,t_1)_\ell(w) \\ z \end{pmatrix} \\ &= \begin{pmatrix} \displaystyle\sum_{\substack{i\in\Upsilon(e-\ell) \\ 1\in i}} x_i + \sum_{\substack{i\in\Upsilon(e-\ell) \\ 1\notin i}} (s_1y_i + \gamma^i_{\ell-1}(z)) \\ \displaystyle\sum_{\substack{i\in\Upsilon(e-\ell) \\ 1\in i}} t_1x_i + \sum_{\substack{i\in\Upsilon(e-\ell) \\ 1\notin i}} y_i \\ z \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

This proves that (x, y, z) is in the image of Φ_ℓ. We have now proved that (3.15) is exact.

Denoting by B the exact complex $0 \to X_e \to X_e \to 0$, we now claim that there is a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{-1}\mathcal{M}_e(X,\alpha) & \longrightarrow & \begin{matrix}\Sigma^{-1}\Delta(1)\\ \oplus \\ \Sigma^{-1}\mathcal{M}_e(X,t\alpha)\end{matrix} & \longrightarrow & \Sigma^{-1}\Delta(t_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_e(X,\alpha) & \longrightarrow & \begin{matrix}\Sigma^{-1}\Delta(1)\\ \oplus \\ \mathcal{N}_e(X,t\alpha)\end{matrix} & \longrightarrow & \Sigma^{-1}\Delta(t_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The columns are exact according to Theorem 3.16 and the top rectangles are readily verified to be commutative. A little diagram chase shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, so that the entire diagram is commutative by construction. Now, the top row is clearly exact, and we have just seen that the middle row is exact, so the exactness of the bottom row follows from the 9-lemma (see, for example, [Eis95, exercise A3.12]) applied in each degree. Thus we have constructed an exact sequence

$$0 \longrightarrow \mathcal{N}_e(X,\alpha) \longrightarrow \begin{matrix}\Sigma^{-1}\Delta(1)\\ \oplus \\ \mathcal{N}_e(X,t\alpha)\end{matrix} \longrightarrow \Sigma^{-1}\Delta(t_1) \longrightarrow 0.$$

in G^R_{e-1}(f,P,S-tor), and since $\Delta(1)$ is exact according to Theorem 3.6(i), equation (3.10) follows. This proves the theorem. □

We can now take the final step in proving that $w_e(X, \alpha)$ is independent of the choice of α. First a lemma.

Lemma 3.24. *If Y is an exact complex in $\mathcal{C}^R_{e+1}$(f,P|S-tor) and β is an S-contraction of Y with weight t, then $w_e(\mathcal{N}_{e+1}(Y,\beta), \eta_{e+1}(Y,\beta)) \in G^R_{e-1}$(f,P|$S$-tor) does not depend on the choice of β (but still depends on the weight t).*

Proof: Let us consider the complex $\tilde{Y} \in \mathcal{C}_e^R(\mathrm{f,P}|S\text{-tor})$, constructed from Y in the way $\tilde{X}$ was constructed from X in Lemma 3.20, and equip $\tilde{Y}$ with the S-contraction $\tilde{\beta}$ induced from β in the sense of Theorem 3.4:

$$0 \longrightarrow \operatorname{im}\partial_e^Y \underset{\partial_e^Y \beta_{e-1}^\nu}{\overset{\partial_e^{\tilde{Y}}}{\rightleftarrows}} Y_{e-1} \underset{\beta_{e-2}^\nu}{\overset{\partial_{e-1}^Y}{\rightleftarrows}} \cdots \rightleftarrows Y_1 \underset{\beta_0^\nu}{\overset{\partial_1^Y}{\rightleftarrows}} Y_0 \longrightarrow 0.$$

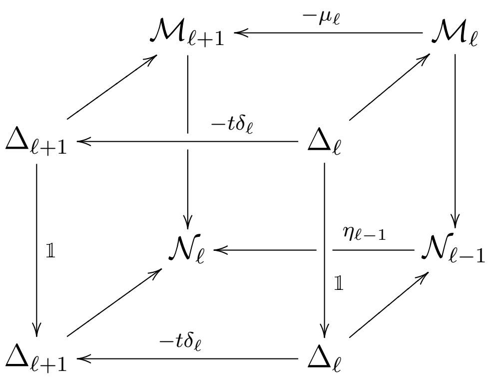
Recall that there is an S-contraction $\delta_{e+1}(Y,t)$ of $\Delta_{e+1}(Y,t)$ with weight t and an S-contraction $\mu_{e+1}(Y,\beta)$ of $\mathcal{M}_{e+1}(Y,\beta)$ with weight t^2. According to Theorem 3.5, the S-contractions $\Sigma^{-1}t\delta_{e+1}(Y,t)$, $\Sigma^{-1}\mu_{e+1}(Y,\beta)$ and $t\beta$ are compatible with the morphisms in the short exact sequence

$$0 \to \Sigma^{-1}\Delta_{e+1}(Y,t) \to \Sigma^{-1}\mathcal{M}_{e+1}(Y,\beta) \to Y \to 0. \tag{3.16}$$

Now, the S-contraction $\eta_{e+1}(Y,\beta)$ on $\mathcal{N}_{e+1}(Y,\beta)$ is induced in the sense of Theorem 3.4 by the S-contraction $\Sigma^{-1}\mu_{e+1}(Y,\beta)$ on $\Sigma^{-1}\mathcal{M}_{e+1}(Y,\beta)$ through the morphism $\Sigma^{-1}\mathcal{M}_{e+1}(Y,\beta) \to \mathcal{N}_{e+1}(Y,\beta)$; similarly, as described above, the S-contraction $\tilde{\beta}$ on $\tilde{Y}$ is induced in the sense of Theorem 3.4 by the S-contraction β on Y through the morphism $Y \to \tilde{Y}$. We claim that this implies that the S-contractions $\Sigma^{-1}t\delta_{e+1}(Y,t)$, $\eta_{e+1}(Y,\beta)$ and $t\tilde{\beta}$ are compatible with the morphisms in the exact sequence

$$0 \to \Sigma^{-1}\Delta_{e+1}(Y,t) \to \mathcal{N}_{e+1}(Y,\beta) \to \tilde{Y} \to 0$$

from (3.8) in Lemma 3.20. This is easy: let $\Delta \overset{\mathrm{def}}{=} \Delta_{e+1}(Y,t)$, $\mathcal{M} \overset{\mathrm{def}}{=} \mathcal{M}_{e+1}(Y,\beta)$, $\mathcal{N} \overset{\mathrm{def}}{=} \mathcal{N}_{e+1}(Y,\beta)$, $\delta \overset{\mathrm{def}}{=} \delta_{e+1}(Y,t)$, $\mu \overset{\mathrm{def}}{=} \mu_{e+1}(Y,\beta)$ and $\eta \overset{\mathrm{def}}{=} \eta_{e+1}(Y,\beta)$. Proving, for example, that $\Sigma^{-1}t\delta$ and η are compatible with the morphism $\Sigma^{-1}\Delta \to \mathcal{N}$ means proving the commutativity of the bottom rectangle of the following diagram for all $\ell \in \mathbb{Z}$ and $\nu = 1,\ldots,d$.



The top rectangle is commutative since $\Sigma^{-1}t\delta$ and $\Sigma^{-1}\mu$ are compatible with the first morphism in (3.16), and the back rectangle is commutative since η is induced from $\Sigma^{-1}\mu$ in the sense of Theorem 3.4. We have constructed the morphism

$\Sigma^{-1}\Delta \to \mathcal{N}$ by inducing it from $\Sigma^{-1}\Delta \to \Sigma^{-1}\mathcal{M}$ via the morphism $\Sigma^{-1}\mathcal{M} \to \mathcal{N}$, so the rectangles on the left and right side must also be commutative. Thus all rectangles except possibly the bottom one are commutative. Since the vertical maps are all surjective, the bottom rectangle now lifts to the top rectangle, and it follows that the bottom rectangle must be commutative. A similar argument shows that η and $t\tilde{\beta}$ are compatible with the morphism $\mathcal{N} \to \tilde{Y}$.

Recalling from Lemma 3.20 that the exactness of Y implies the exactness of $\tilde{Y}$, we now get, using Lemmas 3.19 and 3.20, that

$$\begin{aligned} w_e(\mathcal{N}, \eta) &= w_e(\Sigma^{-1}\Delta, \Sigma^{-1}t\delta) + w_e(\tilde{Y}, t\tilde{\beta}) \\ &= w_{e+1}(\Sigma^{-1}\Delta, \Sigma^{-1}t\delta), \end{aligned}$$

which does not depend on β (but apparently still depends on t). □

Theorem 3.25. *The element $w_e(X, \alpha) \in G^R_{e-1}(\text{f,P}|S\text{-tor})$ does not depend on the choice of α (nor on the weight s): that is, if β is an S-contraction of X with weight t, then $w_e(X, \alpha) = w_e(X, \beta)$.*

Proof: We can assume that the weight s of α equals the weight t of β; for if this is not the case, we consider instead the S-contractions $t\alpha$ and $s\beta$ whose weights are both st, and we know from Theorem 3.23 that $w_e(X, \alpha) = w_e(X, t\alpha)$ and $w_e(X, s\beta) = w_e(X, \beta)$.

Consider the mapping cone $\mathcal{M}(\mathbb{1}_X)$ of the identity morphism $\mathbb{1}_X \colon X \to X$ and the canonical short exact sequence

$$0 \to X \to \mathcal{M}(\mathbb{1}_X) \to \Sigma X \to 0$$

from Theorem 0.4. According to Theorem 3.5, the S-contractions $s\beta$, $\beta * \alpha$ and $\Sigma s\alpha$ all have weight s^2 and are compatible with the morphisms in the above sequence.

Now, the above sequence, which is a sequence in $\mathcal{C}^R_{e+1}(\text{f,P}|S\text{-tor})$, induces by (3.6) from Lemma 3.19 an exact sequence

$$0 \to \mathcal{N}_{e+1}(X, s\beta) \to \mathcal{N}_{e+1}(\mathcal{M}(\mathbb{1}_X), \beta * \alpha) \to \mathcal{N}_{e+1}(\Sigma X, \Sigma s\alpha) \to 0$$

in $\mathcal{C}^R_e(\text{f,P}|S\text{-tor})$. According to the same lemma, the S-contractions $\eta_{e+1}(X, s\beta)$, $\eta_{e+1}(\mathcal{M}(\mathbb{1}_X), \beta * \alpha)$ and $\eta_{e+1}(\Sigma X, \Sigma s\alpha)$, which all have weight s^4, are compatible with the morphisms in the above sequence.

In the construction of $\mathcal{N}_{e+1}(X, s\beta)$ we have considered X as a complex concentrated in degrees $e+1, \dots, 0$. Since X_{e+1} is the zero module, $\Delta_{e+1}(X, s^2)$ is the zero complex, and therefore $\mathcal{N}_{e+1}(X, s\beta) = X$. Furthermore, it is straightforward to see that $\eta_{e+1}(X, s\beta)$ is the same as $s^3\beta$ considered as an S-contraction of X. It now follows from Theorem 3.23 and Lemma 3.19 that

$$\begin{aligned} w_e(X,\beta) &= w_e(X, s^3\beta) \\ &= w_e(\mathcal{N}_{e+1}(X, s\beta), \eta_{e+1}(X, s\beta)) \\ &= w_e(\mathcal{N}_{e+1}(\mathcal{M}(\mathbb{1}_X), \beta * \alpha), \eta_{e+1}(\mathcal{M}(\mathbb{1}_X), \beta * \alpha)) \\ &\quad - w_e(\mathcal{N}_{e+1}(\Sigma X, \Sigma s\alpha), \eta_{e+1}(\Sigma X, \Sigma s\alpha)). \end{aligned}$$

Since $\mathcal{M}(\mathbb{1}_X)$ is exact, Lemma 3.24 implies that the first term in the above difference does not depend on $\beta * \alpha$ and thereby not on β. The second term does not depend on β either, so it follows that the difference depends only on α. Replacing β by α, we therefore find that $w_e(X,\alpha)$ is equal to the same difference, and hence $w_e(X,\alpha) = w_e(X,\beta)$ as desired. □

Definition 3.26. In the light of Theorem 3.25, we shall write $w_e(X)$ to mean $w_e(X,\alpha)$ for any choice of S-contraction α of X.

We have now accomplished the first and hardest task in constructing an inverse to $\iota\colon G^R_{e-1}(\text{f,P}|S\text{-tor}) \to G^R_e(\text{f,P}|S\text{-tor})$. Our second task is achieved in the theorem below. First a lemma.

Lemma 3.27. *Suppose that*

$$0 \longrightarrow \bar{X} \xrightarrow{\bar{\psi}} X \xrightarrow{\psi} \tilde{X} \to 0 \tag{3.17}$$

is an exact sequence in $\mathcal{C}^R_e(\text{f,P}|S\text{-tor})$. *Then a morphism* $\rho\colon \Sigma^{-1}\tilde{X} \to \bar{X}$ *exists such that its mapping cone* $\mathcal{M}(\rho)$ *is isomorphic to* X.

Proof: The existence of the exact sequence in (3.17) implies the existence of an exact sequence of projective modules in each degree. Such a sequence splits, and hence for each $\ell \in \mathbb{Z}$ we can find homomorphisms $\kappa_\ell\colon X_\ell \to \bar{X}_\ell$ and $\tilde{\kappa}\colon \tilde{X}_\ell \to X_\ell$ such that $\bar{\psi}_\ell\kappa_\ell + \tilde{\kappa}_\ell\psi_\ell = \mathbb{1}_{X_\ell}$ (from which it easily follows that $\kappa_\ell\bar{\psi}_\ell = \mathbb{1}_{\bar{X}_\ell}$ and $\psi_\ell\tilde{\kappa}_\ell = \mathbb{1}_{\tilde{X}_\ell}$).

Denote the differentials of $\bar{X}$, X and $\tilde{X}$ by $\bar{\partial}$, ∂ and $\tilde{\partial}$, respectively, and let for each $\ell \in \mathbb{Z}$

$$\rho_\ell = \kappa_\ell\partial_{\ell+1}\tilde{\kappa}_{\ell+1}\colon \tilde{X}_{\ell+1} \to \bar{X}_\ell.$$

We claim that $\rho = (\rho_\ell)_{\ell\in\mathbb{Z}}$ defines a morphism $\Sigma^{-1}\tilde{X} \to \bar{X}$. To prove this we need to show that $\bar{\partial}_\ell\rho_\ell = -\rho_{\ell-1}\tilde{\partial}_{\ell+1}$ for each $\ell \in \mathbb{Z}$. This means showing that $\bar{\partial}_\ell\kappa_\ell\partial_{\ell+1}\tilde{\kappa}_{\ell+1} = -\kappa_{\ell-1}\partial_\ell\tilde{\kappa}_\ell\tilde{\partial}_{\ell+1}$, and since $\bar{\psi}_{\ell-1}$ is injective, this is the same as showing that

$$\bar{\psi}_{\ell-1}\bar{\partial}_\ell\kappa_\ell\partial_{\ell+1}\tilde{\kappa}_{\ell+1} = -\bar{\psi}_{\ell-1}\kappa_{\ell-1}\partial_\ell\tilde{\kappa}_\ell\tilde{\partial}_{\ell+1}. \tag{3.18}$$

The left side of (3.18) is calculated:

$$\begin{aligned}
\bar\psi_{\ell-1}\bar\partial_\ell\kappa_\ell\partial_{\ell+1}\tilde\kappa_{\ell+1} &= \partial_\ell\bar\psi_\ell\kappa_\ell\partial_{\ell+1}\tilde\kappa_{\ell+1} && \text{(since } \bar\psi \text{ is a morphism)}\\
&= \partial_\ell(\mathbb{1}_{X_\ell} - \tilde\kappa_\ell\psi_\ell)\partial_{\ell+1}\tilde\kappa_{\ell+1} && \text{(since } \bar\psi_\ell\kappa_\ell + \tilde\kappa_\ell\psi_\ell = \mathbb{1}_{X_\ell})\\
&= -\partial_\ell\tilde\kappa_\ell\psi_\ell\partial_{\ell+1}\tilde\kappa_{\ell+1} && \text{(since } \partial_\ell\partial_{\ell+1} = 0)\\
&= -\partial_\ell\tilde\kappa_\ell\tilde\partial_{\ell+1}\psi_{\ell+1}\tilde\kappa_{\ell+1} && \text{(since } \psi \text{ is a morphism)}\\
&= -\partial_\ell\tilde\kappa_\ell\tilde\partial_{\ell+1} && \text{(since } \psi_{\ell+1}\tilde\kappa_{\ell+1} = \mathbb{1}_{\tilde X_{\ell+1}}).
\end{aligned}$$

For the right side of (3.18), note that

$$\bar\psi_{\ell-1}\kappa_{\ell-1}\partial_\ell\tilde\kappa_\ell\tilde\partial_{\ell+1} = (\mathbb{1}_{X_{\ell-1}} - \tilde\kappa_{\ell-1}\psi_{\ell-1})\partial_\ell\tilde\kappa_\ell\tilde\partial_{\ell+1},$$

since $\bar\psi_{\ell-1}\kappa_{\ell-1} + \tilde\kappa_{\ell-1}\psi_{\ell-1} = \mathbb{1}_{X_{\ell-1}}$. This again is equal to $\partial_\ell\tilde\kappa_\ell\tilde\partial_{\ell+1}$ because of the calculation

$$\begin{aligned}
\tilde\kappa_{\ell-1}\psi_{\ell-1}\partial_\ell\tilde\kappa_\ell\tilde\partial_{\ell+1} &= \tilde\kappa_{\ell-1}\tilde\partial_\ell\psi_\ell\tilde\kappa_\ell\tilde\partial_{\ell+1} && \text{(since } \psi \text{ is a morphism)}\\
&= \tilde\kappa_{\ell-1}\tilde\partial_\ell\tilde\partial_{\ell+1} && \text{(since } \psi_\ell\tilde\kappa_\ell = \mathbb{1}_{X_\ell})\\
&= 0 && \text{(since } \tilde\partial_\ell\tilde\partial_{\ell+1} = 0).
\end{aligned}$$

This proves that ρ is a morphism of complexes.

Now consider the mapping cone $\mathcal{M}(\rho)$ of ρ. We see that $\mathcal{M}(\rho)$ is the complex

$$\mathcal{M}(\rho) = \quad 0 \longrightarrow \begin{matrix}\bar X_e\\ \oplus\\ \tilde X_e\end{matrix} \xrightarrow{\begin{pmatrix}\bar\partial_e & \rho_{e-1}\\ 0 & \tilde\partial_e\end{pmatrix}} \begin{matrix}\bar X_{e-1}\\ \oplus\\ \tilde X_{e-1}\end{matrix} \longrightarrow \cdots \longrightarrow \begin{matrix}\bar X_0\\ \oplus\\ \tilde X_0\end{matrix} \longrightarrow 0$$

concentrated in degrees $e, \ldots, 0$, and we claim that there is a morphism $\lambda\colon X \to \mathcal{M}(\rho)$ given in degree ℓ by

$$\lambda_\ell = \begin{pmatrix}\kappa_\ell\\ \psi_\ell\end{pmatrix} : X_\ell \to \begin{matrix}\bar X_\ell\\ \oplus\\ \tilde X_\ell\end{matrix} = \mathcal{M}(\rho)_\ell$$

for all $\ell \in \mathbb{Z}$. To see that this in fact defines a morphism of complexes, we need to verify the following equation of matrices.

$$\begin{pmatrix}\bar\partial_\ell & \rho_{\ell-1}\\ 0 & \tilde\partial_\ell\end{pmatrix}\begin{pmatrix}\kappa_\ell\\ \psi_\ell\end{pmatrix} = \begin{pmatrix}\kappa_{\ell-1}\\ \psi_{\ell-1}\end{pmatrix}\partial_\ell.$$

Identity at the $(2,1)$-entry follows from ψ being a morphism, while identity at the $(1,1)$-entry follows from the following calculation

$$\begin{aligned}
\bar\partial_\ell\kappa_\ell + \rho_{\ell-1}\psi_\ell &= \bar\partial_\ell\kappa_\ell + \kappa_{\ell-1}\partial_\ell\tilde\kappa_\ell\psi_\ell && \text{(by definition of } \rho)\\
&= \bar\partial_\ell\kappa_\ell + \kappa_{\ell-1}\partial_\ell(\mathbb{1}_{X_\ell} - \bar\psi_\ell\kappa_\ell) && \text{(since } \bar\psi_\ell\kappa_\ell + \tilde\kappa_\ell\psi_\ell = \mathbb{1}_{X_\ell})\\
&= \bar\partial_\ell\kappa_\ell + \kappa_{\ell-1}\partial_\ell - \kappa_{\ell-1}\bar\psi_{\ell-1}\bar\partial_\ell\kappa_\ell && \text{(since } \bar\psi \text{ is a morphism)}\\
&= \kappa_{\ell-1}\partial_\ell && \text{(since } \kappa_{\ell-1}\bar\psi_{\ell-1} = \mathbb{1}_{\bar X_{\ell-1}}).
\end{aligned}$$

For each $\ell \in \mathbb{Z}$, λ_ℓ must be an isomorphism since it has an inverse map given by

$$\begin{pmatrix}\bar{\psi}_\ell & \tilde{\kappa}_\ell\end{pmatrix} : \begin{matrix}\bar{X}_\ell \\ \oplus \\ \tilde{X}_\ell\end{matrix} \to X_\ell,$$

and it follows that λ must be an isomorphism of complexes. This proves the lemma. □

Theorem 3.28. *The map* $w_e\colon \mathcal{C}_e^R(\text{f,P}|S\text{-tor}) \to G_{e-1}^R(\text{f,P}|S\text{-tor})$ *induces a group homomorphism* $\omega_e\colon G_e^R(\text{f,P}|S\text{-tor}) \to G_{e-1}^R(\text{f,P}|S\text{-tor})$ *defined by* $\omega_e([X]) = w_e(X)$ *for* $X \in \mathcal{C}_e^R(\text{f,P}|S\text{-tor})$.

PROOF: The only thing we need to show is that the relations in $G_e^R(\text{f,P}|S\text{-tor})$ (as described in Definition 2.4) are preserved under the map w_e.

If X is exact, we already know from Lemma 3.20 that $w_e(X) = 0$. Thus, it only remains to show that if

$$0 \longrightarrow \bar{X} \xrightarrow{\bar{\psi}} X \xrightarrow{\psi} \tilde{X} \to 0 \tag{3.19}$$

is an exact sequence in $\mathcal{C}_e^R(\text{f,P}|S\text{-tor})$, then $w_e(X) = w_e(\bar{X}) + w_e(\tilde{X})$. According to Lemma 3.27, the existence of the exact sequence in (3.19) implies the existence of a morphism $\rho\colon \Sigma^{-1}\tilde{X} \to \bar{X}$ with the property that $\mathcal{M}(\rho)$ is isomorphic to X. Now choose S-contractions $\bar{\alpha}$ and $\tilde{\alpha}$ for $\bar{X}$ and $\tilde{X}$ respectively, and let $\bar{s}$ and $\tilde{s}$ denote the weights of $\bar{\alpha}$ and $\tilde{\alpha}$, respectively. Recall that $\bar{\alpha} * \Sigma^{-1}\tilde{\alpha}$ is an S-contraction of $\mathcal{M}(\rho)$ with weight $\bar{s}\tilde{s}$. We now have

$$\begin{aligned} w_e(X) &= w_e(\mathcal{M}(\rho)) \\ &= w_e(\mathcal{M}(\rho), \bar{\alpha} * \Sigma^{-1}\tilde{\alpha}) \\ &= w_e(\bar{X}, \tilde{s}\bar{\alpha}) + w_e(\tilde{X}, \bar{s}\tilde{\alpha}) \\ &= w_e(\bar{X}) + w_e(\tilde{X}), \end{aligned}$$

where the third equality follows from Theorem 3.5 and Lemma 3.19. This proves the theorem. □

We are immediately able to show that our homomorphism ω_e in fact is an isomorphism.

Theorem 3.29. *The group homomorphism*

$$\iota_{e-1}\colon G_{e-1}^R(\text{f,P}|S\text{-tor}) \to G_e^R(\text{f,P}|S\text{-tor})$$

given by $\iota_{e-1}([X]) = [X]$ *is an isomorphism; in fact, the inverse of* ι_{e-1} *is* ω_e.

Proof: If we shift the canonical exact sequence of the mapping cone $\mathcal{M}_e(X, \alpha)$ one degree to the right, we get the exact sequence

$$0 \to \Sigma^{-1}\Delta_e(X, s) \to \Sigma^{-1}\mathcal{M}_e(X, \alpha) \to X \to 0 \tag{3.20}$$

of complexes in $\mathcal{C}_e^R(\text{f,P}|S\text{-tor})$. Theorem 3.16 showed that there is an exact sequence

$$0 \to B \to \Sigma^{-1}\mathcal{M}_e(X, \alpha) \to \mathcal{N}_e(X, \alpha) \to 0 \tag{3.21}$$

in $\mathcal{C}_e^R(\text{f,P}|S\text{-tor})$, where B is the exact complex $0 \to X_e \to X_e \to 0$ concentrated in degrees e and $e-1$. From the exact sequences in (3.20) and (3.21) it now follows that the following holds in $G_e^R(\text{f,P}|S\text{-tor})$.

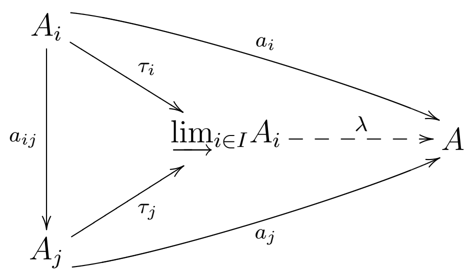
$$\begin{aligned}[X] &= [\Sigma^{-1}\mathcal{M}_e(X, \alpha)] - [\Sigma^{-1}\Delta_e(X, s)] \\ &= [\mathcal{N}_e(X, \alpha)] - [\Sigma^{-1}\Delta_e(X, s)] \\ &= \iota_{e-1}\omega_e([X]).\end{aligned}$$

Suppose now that Y is a complex in $\mathcal{C}_{e-1}^R(\text{f,P}|S\text{-tor})$ and that β is an S-contraction of Y with weight t. Then, considering Y as a complex in $\mathcal{C}_e^R(\text{f,P}|S\text{-tor})$, $\Delta_e(Y, t) = 0$ and $\mathcal{N}_e(Y, \beta) = Y$, and therefore

$$[Y] = [\mathcal{N}_e(Y, \beta)] - [\Sigma^{-1}\Delta_e(Y, t)] = \omega_e\iota_{e-1}([Y]).$$

Thus, ι_{e-1} and ω_e are inverses of each other, and the theorem is proved. □

We are now ready to prove the main theorem. To do so, recall that a *directed set* is a partially ordered set $I = (I, \preccurlyeq)$, satisfying the condition that, for every pair of elements $i, j \in I$, an element $k \in I$ exists such that $i, j \preccurlyeq k$. A *direct system* of Abelian groups is a family $(A_i, a_{ij})_{i \preccurlyeq j}$ of Abelian groups A_i and group homomorphisms $a_{ij}\colon A_i \to A_j$, indexed by pairs of elements $i, j \in I$ with $i \preccurlyeq j$, such that $a_{ii} = \mathbb{1}_{A_i}$ for all $i \in I$ and $a_{jk}a_{ij} = a_{ik}$ for all $i \preccurlyeq j \preccurlyeq k$. Such a direct system of Abelian groups has a *direct limit*, which is an Abelian group $\varinjlim_{i \in I} A_i$ defined as the quotient of $\coprod_{i \in I} A_i$ with the subgroup generated by the elements $m - a_{ij}(m)$ for all $m \in A_i$. The direct limit is uniquely determined (up to isomorphism) by the fact that there are homomorphisms $\tau_i\colon A_i \to \varinjlim_{i \in I} A_i$ for all $i \in I$ with the property that $\tau_j a_{ij} = \tau_i$ for all $i \preccurlyeq j$ and the fact that it is universal with respect to this property, in the sense that if A is another Abelian group with homomorphisms $a_i\colon A_i \to A$ such that $a_j a_{ij} = a_i$ for all $i \preccurlyeq j$, a unique homomorphism $\lambda\colon \varinjlim_{i \in I} A_i \to A$ exists such that $a_i = \lambda\tau_i$ for each $i \in I$:



Theorem 3.30 (main theorem, d > 0). *The group homomorphism*

$$\iota\colon G^R_d(\text{f,P}|S\text{-tor}) \to G^R_\square(\text{f,P}|S\text{-tor})$$

given by $\iota([X]) = [X]$ *is an isomorphism.*

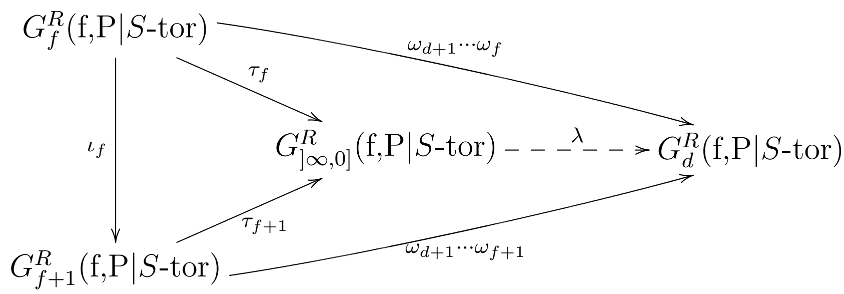
Proof: The sequence

$$G^R_d(\text{f,P}|S\text{-tor}) \xrightarrow{\iota_d} G^R_{d+1}(\text{f,P}|S\text{-tor}) \xrightarrow{\iota_{d+1}} \cdots$$

of Abelian groups $G^R_f(\text{f,P}|S\text{-tor})$ and homomorphisms ι_f for $f \geq d$ is a direct system. It is straightforward to see that the Grothendieck group $G^R_{]\infty,0]}(\text{f,P}|S\text{-tor})$ satisfies the universal property required by a direct limit of the above sequence: the homomorphisms $\tau_f\colon G^R_f(\text{f,P}|S\text{-tor}) \to G^R_{]\infty,0]}(\text{f,P}|S\text{-tor})$ are the maps $[X] \mapsto [X]$, and given a group A and a family $(a_f)_{d\leq f}$ of maps $a_f\colon G^R_f(\text{f,P}|S\text{-tor}) \to A$ as described previously, the map $\lambda\colon G^R_{]\infty,0]}(\text{f,P}|S\text{-tor}) \to A$ is given by $\lambda([X]) = a_f([X])$ for f chosen sufficiently large that $X \in G^R_f(\text{f,P}|S\text{-tor})$. In contrast, since all the homomorphisms ι_f are isomorphisms according to Theorem 3.29, the direct limit must be isomorphic to each of the groups $G^R_f(\text{f,P}|S\text{-tor})$ and τ_f must be an isomorphism for each $f \geq d$.

Now, the homomorphism $\iota_\infty\colon G^R_{]\infty,0]}(\text{f,P}|S\text{-tor}) \to G^R_\square(\text{f,P}|S\text{-tor})$ given by $\iota_\infty([X]) = [X]$ is clearly an isomorphism, the inverse being given by $[X] \mapsto (-1)^n[\Sigma^n X]$ for n chosen sufficiently large that $\Sigma^n X \in \mathcal{C}^R_{]\infty,0]}(\text{f,P}|S\text{-tor})$. Since ι is composed of the isomorphisms $\tau_d\colon G^R_d(\text{f,P}|S\text{-tor}) \to G^R_{]\infty,0]}(\text{f,P}|S\text{-tor})$ and $\iota_\infty\colon G^R_{]\infty,0]}(\text{f,P}|S\text{-tor}) \to G^R_\square(\text{f,P}|S\text{-tor})$, it follows that ι is an isomorphism. □

Substituting A in the above proof with the group $G^R_d(\text{f,P}|S\text{-tor})$ and the a_f's with the isomorphisms $\omega_{d+1}\cdots\omega_f$ (and in particular a_d with the identity map) places us in the situation



where λ must be an isomorphism given by $\lambda([X]) = \omega_{d+1}\cdots\omega_f([X])$ for f chosen sufficiently large that $X \in \mathcal{C}^R_f(\text{f,P}|S\text{-tor})$. The property that the above diagram commutes in the case $f = d$ means that $\lambda\tau_d$ is the identity, and hence λ is inverse to the homomorphism $\tau_d\colon G^R_d(\text{f,P}|S\text{-tor}) \to G^R_{]\infty,0]}(\text{f,P}|S\text{-tor})$. It follows that the inverse of ι is the map $\omega\colon G^R_\square(\text{f,P}|S\text{-tor}) \to G^R_d(\text{f,P}|S\text{-tor})$ given by $\omega([X]) = (-1)^n \omega_{d+1} \cdots \omega_f([\Sigma^n X])$ for n and f chosen sufficiently large that $\Sigma^n X \in \mathcal{C}_f^R(\mathrm{f,P}|S\text{-tor})$.

We still need to prove the main theorem in the case that $d = 0$ where ι is a homomorphism from $G_0^R(\mathrm{f,P}) = K_0(R)$ to $G_\square^R(\mathrm{f,P})$. The only reason that we have required d to be nonnegative so far is to avoid the consideration of "special cases". When $d = 0$, there simply is no S-contraction; however, the technique used previously still works! Given $X \in \mathcal{C}_e^R(\mathrm{f,P})$, the complex $\Delta_e(X)$ in this case is concentrated in degree e and identical to the module X_e, and the morphism $\phi_e(X)\colon X \to \Delta_e(X)$ is given by $\phi_e(X)_e = \mathbb{1}_{X_e}$ and $\phi_e(X)_\ell = 0$ for $\ell \neq e$. Our mapping cone $\mathcal{M}_e(X) = \mathcal{M}(\phi_e(X))$ is then the complex

$$0 \longrightarrow X_e \xrightarrow{\begin{pmatrix} \mathbb{1} \\ -\partial_e^X \end{pmatrix}} \begin{matrix} X_e \\ \oplus \\ X_{e-1} \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -\partial_{e-1}^X \end{pmatrix}} X_{e-2} \xrightarrow{-\partial_{e-2}^X} X_{e-3} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow 0$$

concentrated in degrees $e+1, \ldots, 1$. Removing the identity $X_e \to X_e$ from degrees $e+1$ and e and shifting the result one degree to the right, we obtain the induced complex

$$\mathcal{N}_e(X) = 0 \longrightarrow X_{e-1} \xrightarrow{\partial_{e-1}^X} X_{e-2} \xrightarrow{\partial_{e-2}^X} X_{e-3} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow 0,$$

which is just X without the module in degree e. We now have that

$$[X] = [\mathcal{N}_e(X)] - [\Sigma^{-1}\Delta_e(X)] = [\mathcal{N}_e(X)] + (-1)^e [X_e]$$

in $G_e^R(\mathrm{f,P})$. Repeating the process on the complex $\mathcal{N}_e(X)$, we obtain inductively that $[X] = \sum_{\ell=0}^{e} (-1)^\ell [X_\ell]$ in $G_e^R(\mathrm{f,P})$. The claim is now that the inverse of the homomorphism $\iota\colon G_0^R(\mathrm{f,P}) \to G_\square^R(\mathrm{f,P})$ must be given by taking the alternating sum of the modules in a complex. We state this in a theorem and provide a proof that does not require one to look through the last 25 pages to verify that they indeed make sense in the case $d = 0$.

Theorem 3.31 (main theorem, d = 0). *The group homomorphism*

$$\iota\colon G_0^R(\mathrm{f,P}) \to G_\square^R(\mathrm{f,P})$$

given by $\iota([M]) = [M]$ *is an isomorphism.*

PROOF: We claim that the inverse of ι is the map $\mathcal{A}\colon G_\square^R(\mathrm{f,P}) \to G_0^R(\mathrm{f,P})$ given by $\mathcal{A}([Y]) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell [Y_\ell]$. To see that $\mathcal{A}$ is well defined, note first that, if $0 \to U \to V \to W \to 0$ is a short exact sequence in $\mathcal{C}_\square^R(\mathrm{f,P})$, then there is a short exact sequence of modules in each degree, proving that $\mathcal{A}([V]) = \mathcal{A}([U]) + \mathcal{A}([W])$. Next, if Y is an exact complex in $\mathcal{C}_\square^R(\mathrm{f,P})$, then $\coprod_{\ell \text{ even}} Y_\ell \cong \coprod_{\ell \text{ odd}} Y_\ell$ (see, for example, [Mag02, Corollary 3.47]), proving that $\mathcal{A}([Y]) = 0$. Thus, $\mathcal{A}$ is a well-defined homomorphism.

It is clear that $\mathcal{A} \circ \iota$ is the identity on $G_0^R(\text{f,P})$, so it only remains to verify that $\iota \circ \mathcal{A}$ is the identity on $G_\square^R(\text{f,P})$. Given an arbitrary complex Y in $\mathcal{C}_\square^R(\text{f,P})$, choose e sufficiently large that $Y_\ell = 0$ for $\ell > e$. We then have an exact sequence

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & Y_e & \xrightarrow{\mathbb{1}} & Y_e & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \partial_e^Y} & & \downarrow & & \\
0 & \longrightarrow & Y_{e-1} & \xrightarrow{\mathbb{1}} & Y_{e-1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{e-1}^Y} & & \downarrow{\scriptstyle \partial_{e-1}^Y} & & \downarrow & & \\
0 & \longrightarrow & Y_{e-2} & \xrightarrow{\mathbb{1}} & Y_{e-2} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

of complexes in $\mathcal{C}_\square^R(\text{f,P})$, proving inductively that

$$[Y] = \sum_{\ell \in \mathbb{Z}} [\Sigma^\ell Y_\ell] = \sum_{\ell \in \mathbb{Z}} (-1)^\ell [Y_\ell] = \iota \circ \mathcal{A}([Y])$$

as desired. □

3.4 Consequences of the main theorem

Given all the effort to prove the main theorem, we are pleased to present in this section a collection of results that are established relatively easy as corollaries to the main theorem.

Definition 3.32. If x is an element of R, we let $S(x)$ denote the multiplicative system $\{x^n \,|\, n \in \mathbb{N}_0\}$. If $d \in \mathbb{N}$ and $x = (x_1, \ldots, x_d)$ is a d-tuple of elements, we let $S(x)$ denote the d-tuple $(S(x_1), \ldots, S(x_d))$ of multiplicative systems.

Lemma 3.33. *Suppose that R is Noetherian and local with maximal ideal $\mathfrak{m}$ and that $x = (x_1, \ldots, x_d)$ is a system of parameters (by which d must be equal to the dimension of R). Let M be a finitely generated module. Then* $\operatorname{length}_R M < \infty$ *if and only if M is $S(x)$-torsion.*

Proof: "if": From the assumption it follows that $\operatorname{Ann}_R M \cap S(x_\nu) \neq \emptyset$ for $\nu = 1, \ldots, d$. Thus, we can find $N_1, \ldots, N_d \in \mathbb{N}_0$ such that $x_1^{N_1}, \ldots, x_d^{N_d} \in \operatorname{Ann}_R M$. If $\operatorname{Ann}_R M \subseteq \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal, it therefore follows that $x_1, \ldots, x_d \in \mathfrak{p}$, and since $\dim_R R/\langle x \rangle = 0$ this implies that $\mathfrak{p} = \mathfrak{m}$. Consequently $\operatorname{Supp}_R M \subseteq \{\mathfrak{m}\}$, so $\operatorname{length}_R M < \infty$.

"only if": If M is not $S(x)$-torsion, then $\operatorname{Ann}_R M \cap S(x_\nu) = \emptyset$ for some $\nu \in \{1, \dots, d\}$. Because of the ascending chain condition on R, $\operatorname{Ann}_R M$ can be extended to an ideal $\mathfrak{p}$ maximal with respect to the property of not intersecting $S(x_\nu)$. Such a $\mathfrak{p}$ is prime (see, for example, [Eis95, Proposition 2.11]), and by construction, $\mathfrak{p} \in \operatorname{Supp}_R M$. However, $\mathfrak{p} \neq \mathfrak{m}$ since $x_\nu \notin \mathfrak{p}$, and it follows that $\dim_R M \geq 1$ and thereby that $\operatorname{length}_R M = \infty$. $\square$

Corollary 3.34. *If R is Noetherian and local with $\dim R = d$, then the group homomorphism $\iota\colon G_d^R(\text{f,P}|\text{l}) \to G_\square^R(\text{f,P}|\text{l})$ given by $\iota([X]) = [X]$ is an isomorphism.*

Proof: Letting $x = (x_1, \dots, x_d)$ be a system of parameters, it follows from Lemma 3.33 that $G_\square^R(\text{f,P}|\text{l}) = G_\square^R(\text{f,P}|S(x)\text{-tor})$ and $G_d^R(\text{f,P}|\text{l}) = G_d^R(\text{f,P}|S(x)\text{-tor})$, and from Theorem 3.30 we now get the isomorphism

$$G_d^R(\text{f,P}|\text{l}) = G_d^R(\text{f,P}|S(x)\text{-tor}) \xrightarrow[\iota]{\cong} G_\square^R(\text{f,P}|S(x)\text{-tor}) = G_\square^R(\text{f,P}|\text{l}). \qquad \square$$

Corollary 3.34 shows that at the level of Grothendieck groups, the complexes in $\mathcal{C}_\square^R(\text{f,P}|\text{l})$ can be represented by complexes concentrated in degrees $\dim R, \dots, 0$. In some sense, we cannot do better than this: the *new intersection theorem* (cf. [Rob98, Theorem 13.4.1]) states that, if a complex in $\mathcal{C}_\square^R(\text{f,P}|\text{l})$ is nonexact and concentrated in degrees $n, \dots, 0$, then $n \geq \dim R$.

Lemma 3.35. *Suppose that R is a local Cohen–Macaulay ring of dimension $d > 0$. Then any module M in $\mathcal{C}_0^R(\text{l,pd})$ satisfies the condition that either $M = 0$ or else $\operatorname{pd}_R M = d$. Furthermore, any complex X in $\mathcal{C}_d^R(\text{f,P}|\text{l})$ satisfies the condition that its homology complex $\text{H}(X)$ is concentrated in degree 0: that is, $\text{H}(X)$ is a module in $\mathcal{C}_0^R(\text{l,pd})$.*

Proof: If M is a module in $\mathcal{C}_0^R(\text{l,pd})$, then $M \neq 0$ implies $\operatorname{depth}_R M = 0$ since $\operatorname{depth}_R M \leq \dim_R M = 0$ according to Proposition 0.15 and Theorem 0.10. We therefore get $\operatorname{pd}_R M = \operatorname{depth} R = \dim R = d$, where the first equality follows from the Auslander–Buchsbaum formula (see Theorem 0.16), and the second equality follows from the assumption that R is Cohen–Macaulay.

Now let X be a complex in $\mathcal{C}_d^R(\text{f,P}|\text{l})$. The modules in X are free, so for $\ell = 0, \dots, d$ we either have $X_\ell = 0$ and thereby $\operatorname{depth}_R X_\ell = \infty$, or $X_\ell \neq 0$ and thereby $\operatorname{depth}_R X_\ell = \operatorname{depth} R = \dim R = d$. If $\text{H}(X)$ is not concentrated in degree 0, it therefore follows from the acyclicity lemma (see Lemma 0.17) that $1 \leq \operatorname{depth}_R \text{H}_\ell(X) < \infty$ for some $\ell > 0$. However, according to Proposition 0.15 and Theorem 0.10, we also have $\operatorname{depth}_R \text{H}_\ell(X) \leq \dim_R \text{H}_\ell(X) = 0$, since X has homologies of finite length. This is a contradiction, so X must be concentrated in degree 0. $\square$

Lemma 3.35 shows that we are in a position where we can apply Theorem 2.11 to obtain a homomorphism $\mathcal{H}\colon G_d^R(\text{f,P}|\text{l}) \to G_0^R(\text{l,pd})$ given by $\mathcal{H}([X]) = [\text{H}(X)]$.

Further, the requirements of Theorem 2.12 are satisfied, so there is a homomorphism $\mathcal{R}\colon G_0^R(\text{l,pd}) \to G_\square^R(\text{f,P}|\text{l})$ given by $\mathcal{R}([M]) = [X]$, where $X \in \mathcal{C}_\square^R(\text{f,P}|\text{l})$ is a projective resolution of M. But $G_d^R(\text{f,P}|\text{l})$ is isomorphic to $G_\square^R(\text{f,P}|\text{l})$ according to Corollary 3.34, and the following diagram commutes.

$$\begin{array}{ccc} G_d^R(\text{f,P}|\text{l}) & \xrightarrow[\cong]{\iota} & G_\square^R(\text{f,P}|\text{l}) \\ {\scriptstyle\mathcal{H}}\big\downarrow & \nearrow{\scriptstyle\mathcal{R}} & \\ G_0^R(\text{l,pd}) & & \end{array}$$

As one could hope for, it turns out that $\mathcal{H}$ and $\mathcal{R}$ are isomorphisms too.

Corollary 3.36. *If R is a local Cohen-Macaulay ring of dimension $d > 0$, then the group homomorphism $\mathcal{H}\colon G_d^R(\text{f,P}|\text{l}) \to G_0^R(\text{l,pd})$ from Theorem 2.11 is an isomorphism, and so is the group homomorphism $\mathcal{R}\colon G_0^R(\text{l,pd}) \to G_\square^R(\text{f,P}|\text{l})$ from Theorem 2.12. In particular, there are isomorphisms*

$$G_d^R(\text{f,P}|\text{l}) \cong G_0^R(\text{l,pd}) \cong G_\square^R(\text{f,P}|\text{l}).$$

Proof: We have already observed that the homomorphisms involved are well defined, and that $\mathcal{R} \circ \mathcal{H} = \iota$ is an isomorphism. Thus, we already know that $\mathcal{H}$ is injective and $\mathcal{R}$ is surjective. Now, if M is a module in $\mathcal{C}_0^R(\text{l,pd})$, according to Lemma 3.35, M has a projective resolution X in $\mathcal{C}_d^R(\text{f,P}|\text{l})$, proving that $[M] = \mathcal{H}([X])$ and thereby that $\mathcal{H}$ is surjective. Consequently $\mathcal{H}$ is an isomorphism, and it follows that $\mathcal{R}$ is an isomorphism as well. $\square$

Lemma 3.37. *Suppose that R is Noetherian and let $x = (x_1, \dots, x_d)$ be a regular sequence of length $d > 0$. Then any complex X in $\mathcal{C}_d^R(\text{f,P}|S(x)\text{-tor})$ satisfies the condition that its homology complex $\mathrm{H}(X)$ is concentrated in degree 0: that is, $\mathrm{H}(X)$ is a module in $\mathcal{C}_0^R(\text{f,pd},S(x)\text{-tor})$.*

Proof: Let X be a nonexact complex in $\mathcal{C}_d^R(\text{f,P}|S(x)\text{-tor})$ and let t denote the largest integer such that $\mathrm{H}_t(X) \neq 0$; this exists since $\mathrm{H}(X) \neq 0$ and X is bounded. We already know that $t \geq 0$, so let us assume that $t > 0$ and try to reach a contradiction.

Let $\mathfrak{p}$ be an associated prime of $\mathrm{H}_t(X)$. Since $\mathrm{H}(X)$ is $S(x)$-torsion, we can find $N_1, \dots, N_d \in \mathbb{N}$ such that $x_1^{N_1}, \dots, x_d^{N_d} \in \operatorname{Ann}_R \mathrm{H}_t(X) \subseteq \mathfrak{p}$ and thereby $x_1, \dots, x_d \in \mathfrak{p}$. Consequently, $(x_1/1, \dots, x_d/1)$ is an $R_\mathfrak{p}$-sequence in $\mathfrak{p}_\mathfrak{p}$ (cf. [BH93, Corollary 1.1.3(i)]), so $\operatorname{depth} R_\mathfrak{p} \geq d > 0$.

Now, localization preserves exactness and projectivity of modules, so the projective resolution

$$0 \to X_d \to \cdots \to X_{t+1} \to \operatorname{im} \partial_{t+1}^X \to 0$$

of $\operatorname{im}\partial^X_{t+1}$ induces a projective resolution

$$0 \to (X_d)_{\mathfrak{p}} \to \cdots \to (X_{t+1})_{\mathfrak{p}} \to (\operatorname{im}\partial^X_{t+1})_{\mathfrak{p}} \to 0$$

of $(\operatorname{im}\partial^X_{t+1})_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module, showing that $\operatorname{pd}_{R_{\mathfrak{p}}}(\operatorname{im}\partial^X_{t+1})_{\mathfrak{p}} \leq d-(t+1)$. From the Auslander–Buchsbaum formula (Theorem 0.16), it now follows that

$$\operatorname{depth}_{R_{\mathfrak{p}}}(\operatorname{im}\partial^X_{t+1})_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{pd}_{R_{\mathfrak{p}}}(\operatorname{im}\partial^X_{t+1})_{\mathfrak{p}} \geq t+1 \geq 2$$

if $\operatorname{im}\partial^X_{t+1}$ is nontrivial, whereas $\operatorname{depth}_{R_{\mathfrak{p}}}(\operatorname{im}\partial^X_{t+1})_{\mathfrak{p}} = \infty$ if $\operatorname{im}\partial^X_{t+1}$ is trivial; in either case, $\operatorname{depth}_{R_{\mathfrak{p}}}(\operatorname{im}\partial^X_{t+1})_{\mathfrak{p}} \geq 2$.

According to Proposition 0.18(i), $\operatorname{depth}_{R_{\mathfrak{p}}}(\ker\partial^X_t)_{\mathfrak{p}} \geq 1$, since $(\ker\partial^X_t)_{\mathfrak{p}}$ is a submodule of the nontrivial free $R_{\mathfrak{p}}$-module $(X_t)_{\mathfrak{p}}$ that has $\operatorname{depth}_{R_{\mathfrak{p}}}(X_t)_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} \geq d \geq 1$. From Proposition 0.18(iii) applied to the short exact sequence

$$0 \to (\operatorname{im}\partial^X_{t+1})_{\mathfrak{p}} \to (\ker\partial^X_t)_{\mathfrak{p}} \to (\mathrm{H}_t(X))_{\mathfrak{p}} \to 0,$$

it now follows that $\operatorname{depth}_{R_{\mathfrak{p}}}(\mathrm{H}_t(X))_{\mathfrak{p}} \geq 1$. This is a contradiction, however, because $\operatorname{depth}_{R_{\mathfrak{p}}}(\mathrm{H}_t(X))_{\mathfrak{p}} = 0$, since $\mathfrak{p}$ is associated to $\mathrm{H}_t(X)$. Thus, $t=0$ as desired. □

Once again, we are in position to apply Theorem 2.11 as well as Theorem 2.12, providing us with homomorphisms $\mathcal{H}\colon G^R_d(\text{f,P}|S(x)\text{-tor}) \to G^R_0(\text{f,pd},S(x)\text{-tor})$ and $\mathcal{R}\colon G^R_0(\text{f,pd},S(x)\text{-tor}) \to G^R_{\square}(\text{f,P}|S(x)\text{-tor})$ that compose by Theorem 3.30 to an isomorphism $\iota\colon G^R_d(\text{f,P}|S(x)\text{-tor}) \to G^R_{\square}(\text{f,P}|S(x)\text{-tor})$ as illustrated by the following commutative diagram.

$$\begin{array}{ccc} G^R_d(\text{f,P}|S(x)\text{-tor}) & \xrightarrow[\cong]{\iota} & G^R_{\square}(\text{f,P}|S(x)\text{-tor}) \\ \downarrow{\scriptstyle\mathcal{H}} & \nearrow{\scriptstyle\mathcal{R}} & \\ G^R_0(\text{f,pd},S(x)\text{-tor}) & & \end{array}$$

According to Theorem 3.31 this even holds when $d=0$. Regardless of whether $d=0$ or not, $\mathcal{H}$ and $\mathcal{R}$ once again turn out to be isomorphisms.

Corollary 3.38. *If R is Noetherian and local, and $x=(x_1,\ldots,x_d)$ is a regular sequence of length $d \geq 0$, then the group homomorphism*

$$\mathcal{H}\colon G^R_d(\text{f,P}|S(x)\text{-tor}) \to G^R_0(\text{f,pd},S(x)\text{-tor})$$

from Theorem 2.11 is an isomorphism, and so is the group homomorphism

$$\mathcal{R}\colon G^R_0(\text{f,pd},S(x)\text{-tor}) \to G^R_{\square}(\text{f,P}|S(x)\text{-tor})$$

from Theorem 2.12. In particular, there are isomorphisms

$$G^R_d(\text{f,P}|S(x)\text{-tor}) \cong G^R_0(\text{f,pd},S(x)\text{-tor}) \cong G^R_{\square}(\text{f,P}|S(x)\text{-tor}).$$

PROOF: We already know that the homomorphisms involved are well defined and that $\mathcal{R} \circ \mathcal{H} = \iota$ is an isomorphism. Thus, it suffices to show that $\mathcal{H}$ is surjective.

So let M be a module in $\mathcal{C}_0^R(\text{f,pd},S(x)\text{-tor})$, and let us show by induction on $p = \operatorname{pd}_R M$ that $[M] \in \operatorname{im}\mathcal{H}$. If $p \leq d$, it is clear that $[M] \in \operatorname{im}\mathcal{H}$ since, in this case, M has a projective resolution in $\mathcal{C}_d^R(\text{f,P}|S(x)\text{-tor})$, so assume that $p > d$. Choose a finitely generated free module F and a surjective homomorphism $f\colon F \to M$. Next, using the fact that M is $S(x)$-torsion, choose $N_1, \dots, N_d \in \mathbb{N}$ so that $x_1^{N_1}, \dots, x_d^{N_d} \in \operatorname{Ann}_R M$, and let $\bar{F} = F/\langle x_1^{N_1}, \dots, x_d^{N_d}\rangle F$. The surjection f induces a surjection $\bar{f}\colon \bar{F} \to M$. Letting K denote the kernel of $\bar{f}$, we then have an exact sequence

$$0 \to K \to \bar{F} \to M \to 0.$$

Now, $\bar{F}$ is the direct sum of copies of $R/\langle x_1^{N_1}, \dots, x_d^{N_d}\rangle$, so it follows from Theorems 0.13 and 3.6(iii) that $\operatorname{pd}_R \bar{F} = d$, and Theorem 0.7 therefore gives $\operatorname{pd}_R K = \operatorname{pd}_R \bar{F} - 1 = d - 1$. By construction, $\bar{F}$ and K are $S(x)$-torsion, so $\bar{F}$ and K are modules in $\mathcal{C}_0^R(\text{f,pd},S(x)\text{-tor})$, and the induction hypothesis yields $[M] = [\bar{F}] - [K] \in \operatorname{im}\mathcal{H}$. Consequently $\mathcal{H}$ is surjective, and it follows that $\mathcal{H}$ and $\mathcal{R}$ are isomorphisms. □

Note that the case $d = 0$ simply states that

$$K_0(R) = G_0^R(\text{f,P}) \cong G_0^R(\text{f,pd}) \cong G_\square^R(\text{f,P}),$$

and that we do not need R to be local to prove this. However, if R is local, Example 2.14 reveals that these groups are all isomorphic to $\mathbb{Z}$ through the rank on $G_0^R(\text{f,P})$. The proof of Theorem 3.31 shows that the isomorphism $G_0^R(\text{f,pd}) \cong \mathbb{Z}$ is given by taking $[M] \in G_0^R(\text{f,pd})$ to the alternating sum of the ranks of the modules in a bounded, finite, free resolution of M; in other words, the Euler characteristic $\chi^R(-)\colon G_0^R(\text{f,pd}) \to \mathbb{Z}$ is an isomorphism. It follows that, in $G_0^R(\text{f,pd})$, we have $[M] = \chi^R(M)[R]$. We shall be needing this fact later, so let us state it as a corollary.

Corollary 3.39. *If R is Noetherian and local, and M is a module in $\mathcal{C}_0^R(\text{f,pd})$, then $[M] = \chi^R(M)[R]$ in $G_0^R(\text{f,pd})$.* □

The proofs of Lemma 3.37 and Corollary 3.38 in the case $d = 1$ clearly show that the multiplicative systems $S(x) = S(x_1) = \{x_1^n \mid n \in \mathbb{N}_0\}$ can be replaced by *any* multiplicative system S containing only non-zerodivisors: that is, any multiplicative system S with $S \cap \operatorname{Zd} R = \emptyset$. This is because any element of such a multiplicative system will itself constitute a regular sequence of length 1. Consequently, we can improve Corollary 3.38 slightly in the case $d = 1$.

Corollary 3.40. *Suppose R is Noetherian and S is a multiplicative system with $S \cap \operatorname{Zd} R = \emptyset$. Then the group homomorphism*

$$\mathcal{H}\colon G_1^R(\text{f,P}|S\text{-tor}) \to G_0^R(\text{f},S\text{-tor})$$

from Theorem 2.11 is an isomorphism, and so is the group homomorphism

$$\mathcal{R}\colon G_0^R(\text{f,pd,}S\text{-tor}) \to G_\square^R(\text{f,P}|S\text{-tor})$$

from Theorem 2.12. In particular, there are isomorphisms

$$G_1^R(\text{f,P}|S\text{-tor}) \cong G_0^R(\text{f,pd,}S\text{-tor}) \cong G_\square^R(\text{f,P}|S\text{-tor}). \qquad \square$$

For the next corollary, note that the Grothendieck groups $G_d^R(\text{f,pd}|\text{gr} \geq d)$ and $G_0^R(\text{f,pd,gr} \geq d)$ for $d \geq 1$ satisfy the conditions of Theorem 2.11; for if X is a complex in $\mathcal{C}_d^R(\text{f,P}|\text{gr} \geq d)$, we can, using Proposition 0.19(iii), find a regular sequence $x = (x_1, \ldots, x_d)$ of length d contained in the annihilator of *all* the homology modules of X. Then X will be homologically $S(x)$-torsion, and it follows from Lemma 3.37 that the homology of X is concentrated in degree 0. Consequently, we can consider the homomorphism $\mathcal{H}\colon G_d^R(\text{f,P}|\text{gr} \geq d) \to G_0^R(\text{f,pd,gr} \geq d)$ from Theorem 2.11. Note also that the Grothendieck groups $G_0^R(\text{f,pd,gr} \geq d)$ and $G_\square^R(\text{f,P}|\text{gr} \geq d)$ satisfy the conditions of Theorem 2.12, thereby allowing us to consider the homomorphism $\mathcal{R}\colon G_0^R(\text{f,pd,gr} \geq d) \to G_\square^R(\text{f,P}|\text{gr} \geq d)$. In all that has just been said, we can replace $G_0^R(\text{f,pd,gr} \geq d)$ by $G_0^R(\text{f,}d\text{-perf})$, thereby obtaining a homology isomorphism $\mathcal{H}'\colon G_d^R(\text{f,P}|\text{gr} \geq d) \to G_0^R(\text{f,}d\text{-perf})$ and a homomorphism $\mathcal{R}'\colon G_0^R(\text{f,}d\text{-perf}) \to G_\square^R(\text{f,P}|\text{gr} \geq d)$. There is also the natural homomorphism $\iota'\colon G_0^R(\text{f,}d\text{-perf}) \to G_0^R(\text{f,gr} \geq d)$ induced by the inclusion of the underlying categories, so the situation is as in the following commutative diagram.

$$\begin{array}{ccccc} & & G_0^R(\text{f,}d\text{-perf}) & & \\ & \overset{\mathcal{H}'}{\nearrow} & & \overset{\mathcal{R}'}{\searrow} & \\ G_d^R(\text{f,P}|\text{gr} \geq d) & & \downarrow \iota' & & G_\square^R(\text{f,P}|\text{gr} \geq d) \\ & \underset{\mathcal{H}}{\searrow} & & \underset{\mathcal{R}}{\nearrow} & \\ & & G_0^R(\text{f,pd,gr} \geq d) & & \end{array} \qquad (3.22)$$

As one could hope, all the homomorphisms in the diagram turn out to be isomorphisms.

Corollary 3.41. *If R is Noetherian and local, and $d \geq 1$, then the group homomorphisms*

$$\mathcal{H}\colon G_d^R(\text{f,P}|\text{gr} \geq d) \to G_0^R(\text{f,pd,gr} \geq d)$$

and

$$\mathcal{H}'\colon G_d^R(\text{f,P}|\text{gr} \geq d) \to G_0^R(\text{f,}d\text{-perf})$$

from Theorem 2.11 are isomorphisms, and so are the group homomorphisms

$$\mathcal{R}\colon G_0^R(\text{f,pd,gr} \geq d) \to G_\square^R(\text{f,P}|\text{gr} \geq d)$$

and

$$\mathcal{R}'\colon G_0^R(\text{f},d\text{-perf}) \to G_\square^R(\text{f,P}|\text{gr} \geq d)$$

from Theorem 2.12. In particular, there are isomorphisms

$$G_d^R(\text{f,P}|\text{gr} \geq d) \cong G_0^R(\text{f},d\text{-perf}) \cong G_0^R(\text{f,pd,gr} \geq d) \cong G_\square^R(\text{f,P}|\text{gr} \ \geq d).$$

PROOF: If d is so large that there are no regular sequences in R of length d, then the involved Grothendieck groups are all trivial and the theorem holds. We can therefore assume that regular sequences of length d do exist. Let us define an equivalence relation on the set of such sequences, letting a regular sequence $x = (x_1, \ldots, x_d)$ be equivalent to a regular sequence $x' = (x'_1, \ldots, x'_d)$ whenever

$$\text{Rad}_R\langle x_1, \ldots, x_d\rangle = \text{Rad}_R\langle x'_1, \ldots, x'_d\rangle.$$

It is clear that this, indeed, is an equivalence relation. Denote the set of equivalence classes by E, and let us partially order E by reversed inclusion of radical ideals: that is,

$$x \preccurlyeq x' \overset{\text{def}}{\iff} \text{Rad}_R\langle x_1, \ldots, x_d\rangle \supseteq \text{Rad}_R\langle x'_1, \ldots, x'_d\rangle$$

for $x, x' \in E$. (It is of course the *equivalence classes* of x and x' that belong to E, but this unimportant technicality will be ignored here.) $E = (E, \preccurlyeq)$ is a directed set, for if x and x' are regular sequences of length d, then according to Proposition 0.19(iii), we can find a regular sequence x'' of length d contained in $\langle x\rangle \cap \langle x'\rangle$ and hence satisfying the condition that $x, x' \preccurlyeq x''$.

Now, the category $\mathcal{C}_0^R(\text{f,pd},S(x)\text{-tor})$ is uniquely determined by the equivalence class of x in E, since for any finitely generated module M,

$$\begin{aligned} M \text{ is } S(x)\text{-torsion} &\iff \forall \nu \in \{1, \ldots, d\} \exists N_\nu \in \mathbb{N}_0 \colon x_\nu^{N_\nu} \in \text{Ann}_R M \\ &\iff \langle x_1, \ldots, x_d\rangle \subseteq \text{Rad}_R(\text{Ann}_R M) \\ &\iff \text{Rad}_R\langle x_1, \ldots, x_d\rangle \subseteq \text{Rad}_R(\text{Ann}_R M). \end{aligned}$$

Thus, we can consider the family of Grothendieck groups $G_0^R(\text{f,pd},S(x)\text{-tor})$ indexed by the equivalence classes in E. Given $x, x' \in E$ with $x \preccurlyeq x'$, there is a homomorphism

$$\iota_{x,x'}\colon G_0^R(\text{f,pd},S(x)\text{-tor}) \to G_0^R(\text{f,pd},S(x')\text{-tor})$$

given by $\iota_{x,x'}([M]) = [M]$; this is well defined, as seen from the bi-implications above. Consequently $(G_0^R(\text{f,pd},S(x)\text{-tor}), \iota_{x,x'})_{x\preccurlyeq x'}$ is a direct system, and it follows that it has a direct limit $\varinjlim_{x\in E} G_0^R(\text{f,pd},S(x)\text{-tor})$ equipped with homomorphisms

$$\tau_x\colon G_0^R(\text{f,pd},S(x)\text{-tor}) \to \varinjlim_{x\in E} G_0^R(\text{f,pd},S(x)\text{-tor})$$

for each $x \in E$.

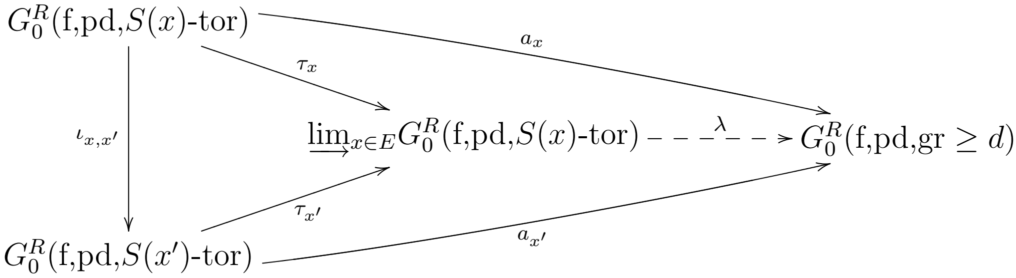
If $x = (x_1, \ldots, x_d)$ is a regular sequence and M is a finitely generated $S(x)$-torsion module, we can find $N_1, \ldots, N_d \in \mathbb{N}$ such that $x_1^{N_1}, \ldots, x_d^{N_d} \in \operatorname{Ann}_R M$, and it follows from Theorem 0.13 that $\operatorname{grade}_R M \geq d$. Thus, there are natural homomorphisms

$$a_x \colon G_0^R(\text{f,pd,}S(x)\text{-tor}) \to G_0^R(\text{f,pd,gr} \geq d)$$

given by $a_x([M]) = [M]$. Since these commute with the homomorphisms $\iota_{x,x'}$, the universal property of the direct limit provides us with a homomorphism

$$\lambda \colon \varinjlim_{x \in E} G_0^R(\text{f,pd,}S(x)\text{-tor}) \to G_0^R(\text{f,pd,gr} \geq d)$$

such that $a_x = \lambda\tau_x$ for all $x \in E$:



We claim that λ is an isomorphism. Surjectivity is clear, for if M is a module in $\mathcal{C}_0^R(\text{f,pd,gr} \geq d)$, we can find a regular sequence $x = (x_1, \ldots, x_d)$ of length d in $\operatorname{Ann}_R M$, and hence M lies in $\mathcal{C}_0^R(\text{f,pd,}S(x)\text{-tor})$ and $[M] = a_x([M]) = \lambda\tau_x([M])$ in $G_0^R(\text{f,pd,gr} \geq d)$. To see that λ is injective, note that, since the image of the maps τ_x span the direct limit (as can be seen from its concrete construction), it suffices to prove that the assumption $[M] = [M']$ in $G_0^R(\text{f,pd,gr} \geq d)$ leads to $[M] = [M']$ in some $G_0^R(\text{f,pd,}S(x)\text{-tor})$. So suppose that $[M] = [M']$ in $G_0^R(\text{f,pd,gr} \geq d)$. It then follows from Proposition 2.6 that we can find modules U, V, V' and W in $\mathcal{C}_0^R(\text{f,pd,gr} \geq d)$ such that there are exact sequences

$$0 \to U \to V \to W \to 0 \quad \text{and} \quad 0 \to U \to V' \to W \to 0,$$

and such that $M \oplus V \cong M' \oplus V'$. Using Proposition 0.19(iii) repeatedly, we can find a regular sequence $x = (x_1, \ldots, x_d)$ of length d contained in the annihilators of all the modules M, M', U, V, V' and W, and it follows that $[M] = [M']$ in $G_0^R(\text{f,pd,}S(x)\text{-tor})$. Consequently, λ is injective.

We have now shown that $G_0^R(\text{f,pd,gr} \geq d)$ is the direct limit of the direct system $(G_0^R(\text{f,pd,}S(x)\text{-tor}), \iota_{x,x'})_{x \preccurlyeq x'}$. By the same methods one can show that $G_d^R(\text{f,P}|\text{gr} \geq d)$ and $G_\square^R(\text{f,P}|\text{gr} \geq d)$ are the direct limits of the direct systems $(G_d^R(\text{f,P}|S(x)\text{-tor}), \iota_{x,x'})_{x \preccurlyeq x'}$ and $(G_\square^R(\text{f,P}|S(x)\text{-tor}), \iota_{x,x'})_{x \preccurlyeq x'}$ respectively, where the homomorphisms $\iota_{x,x'}$ now are given by $\iota_{x,x'}([X]) = [X]$; getting this far requires applying Proposition 0.19(iii) repeatedly to obtain a regular sequence contained in the annihilator of *all* the homology modules of a complex.

Nevertheless, we already know from Corollary 3.38 that there are isomorphisms

$$G_d^R(\text{f,P}|S(x)\text{-tor}) \cong G_0^R(\text{f,pd,}S(x)\text{-tor}) \cong G_\square^R(\text{f,P}|S(x)\text{-tor})$$

for all regular sequences x of length d, the first of these being given by the homomorphism $\mathcal{H}_x$ from Theorem 2.11 and the second being given by the homomorphism $\mathcal{R}_x$ from Theorem 2.12. Hence there must also be isomorphisms

$$G_d^R(\text{f,P}|\text{gr} \geq d) \cong G_0^R(\text{f,pd,gr} \geq d) \cong G_\square^R(\text{f,P}|\text{gr} \geq d)$$

between the direct limits, the first of these being given by the homomorphism $\mathcal{H}$ from Theorem 2.11 and the second being given by the homomorphism $\mathcal{R}$ from Theorem 2.12. In particular, $\mathcal{R}$ and $\mathcal{H}$ are isomorphisms.

Using the commutativity of diagram (3.22), it now follows that $\mathcal{R}'\mathcal{H}'$ is an isomorphism. To show that $\mathcal{R}'$ as well as $\mathcal{H}'$ are isomorphisms, it therefore suffices to show that $\mathcal{H}'$ is surjective. This, however, is clear since any finitely generated d-perfect module by definition has a projective resolution in $\mathcal{C}_d^R(\text{f,P}|\text{gr} \geq d)$. Using the commutativity of diagram (3.22) once more, it follows that ι' also is an isomorphism. □

Note that Corollary 3.41 actually holds when $d = 0$, and that including this case is unnecessary, as it is already part of Corollary 3.38.

Chapter 4

Groups of matrices: K_1

Chapter 3 introduced the zeroth algebraic K-group of R as a special instance of the more general concept of Grothendieck groups. Likewise, the first algebraic K-group of R is a special instance of the more general concept of *Bass–Whitehead groups*. However, in this thesis the full generality is not needed, so we discuss here only the traditional first algebraic K-group $K_1(R)$.

4.1 The first algebraic K-group

If $m, n \in \mathbb{N}$, the set $R^{m\times n}$ of $m \times n$ matrices is a free R-module with a basis consisting of the *matrix units* ϵ_{ij} having (i, j)-entry equal to 1 and all other entries equal to 0. Considering only square matrices, we obtain the ring $M_n(R)$ of $n \times n$ matrices over R, in which the matrix units multiply according to the rule

$$\epsilon_{ij}\epsilon_{kl} = \begin{cases} \epsilon_{il}, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

The *general linear group* $GL_n(R)$ is the multiplicative group consisting of the invertible elements of $M_n(R)$: that is, the matrices whose determinants lie in R^*. The neutral element in $GL_n(R)$ is the identity matrix $I_n = \epsilon_{11} + \cdots + \epsilon_{nn}$.

One way to determine whether a given $n \times n$ matrix A is invertible or not is to apply the row operations in the Gauss–Jordan elimination process and see whether the identity matrix can be obtained. There are three kinds of row operations, and each of these corresponds to left multiplication by an invertible matrix:

(i) adding $r \in R$ times the j'th row to the i'th row for $i \neq j$, which corresponds to left multiplication by $e_{ij}(r) = I_n + r\epsilon_{ij}$;

(ii) interchanging the j'th row with the i'th row, which corresponds to left multiplication by $p_{ij} = I_n - \epsilon_{ii} - \epsilon_{jj} + \epsilon_{ij} + \epsilon_{ji}$; and

(iii) multiplying the i'th row by a unit $u \in R^*$, which corresponds to left multiplication by $d_i(u) = I_n + (u - 1)\epsilon_{ii}$.

The row operations in (i) are called *elementary row transvections*, and the subgroup of $GL_n(R)$ that they generate is denoted by $E_n(R)$ and is referred to as the group of *elementary matrices of degree n*. Since $e_{ij}(r)^{-1} = e_{ij}(-r)$, $E_n(R)$ consists of finite products of elementary row transvections. The subgroup of $GL_n(R)$ generated by the matrices p_{ij} from (ii) is denoted by $P_n(R)$, and it consists of the *permutation matrices*

$$P_\sigma = \epsilon_{\sigma(1)1} + \cdots + \epsilon_{\sigma(n)n}$$

for $\sigma \in S_n$, where S_n is the symmetric group of degree n consisting of all permutations of the set $\{1, \ldots, n\}$. The subgroup of $GL_n(R)$ generated by the matrices $d_i(u)$ from (iii) is denoted by $D_n(R)$, and it consists of the *invertible diagonal matrices*

$$\mathrm{diag}(u_1, \ldots, u_n) = u_1\epsilon_{11} + \cdots + u_n\epsilon_{nn}$$

for $u_1, \ldots, u_n \in R^*$.

The determinants of the three types of row operations are $\det_R(e_{ij}(r)) = 1$, $\det_R(p_{ij}) = -1$ and $\det_R(d_i(u)) = u$. In particular the matrices of $E_n(R)$ all have determinant 1, so performing elementary row transvections on a matrix does not change its determinant. $A, B \in M_n(R)$ are said to be *t-row equivalent* if B can be obtained from A by elementary row transvections: that is, if $B \in E_n(R)A$. This is an equivalence relation. Whenever A is invertible, so is its entire t-row equivalence class, and hence t-row equivalence on $M_n(R)$ restricts to an equivalence relation on $GL_n(R)$.

For the next theorem, recall that if G is a group and H is a subset of G, $H \lhd G$ means that H is a normal subgroup of G. If H and K are subgroups of G, we say that K *normalizes* H if $xHx^{-1} \subseteq H$ for all $x \in K$. In this case, it follows that $HK = KH$ is a subgroup of G and that $H \lhd HK$.

Theorem 4.1. *For all $n \in \mathbb{N}$, $P_n(R)$ normalizes $D_n(R)$ and $D_n(R)$ normalizes $E_n(R)$, and hence*

$$ML_n(R) \stackrel{\text{def}}{=} D_n(R)P_n(R) \quad \text{and} \quad GE_n(R) \stackrel{\text{def}}{=} D_n(R)E_n(R)$$

are subgroups of $GL_n(R)$ with $D_n(R) \lhd ML_n(R)$ and $E_n(R) \lhd GE_n(R)$.

Proof: The subgroup $P_n(R)$ is generated by the matrices p_{ij}. The inverse of p_{ij} is p_{ij}, so given such a matrix and a matrix $\mathrm{diag}(u_1, \ldots, u_n) \in D_n(R)$, we need to show that $p_{ij}d_k(u)p_{ij}$ is an invertible diagonal matrix. Multiplication on the left by p_{ij} interchanges rows i and j, and multiplication on the right by p_{ij} interchanges columns i and j, and hence taking $\mathrm{diag}(u_1, \ldots, u_n)$ to $p_{ij}\,\mathrm{diag}(u_1, \ldots, u_n)p_{ij}$ interchanges the (i, i)-entry with the (j, j)-entry, and the resulting matrix is an invertible diagonal matrix. Thus, $P_n(R)$ normalizes $D_n(R)$.

The subgroup $D_n(R)$ is generated by the matrices $d_i(u)$ for $u \in R^*$. So given such a matrix and an elementary row transvection $e_{jk}(r)$, it suffices to show that

$d_i(u)e_{jk}(r)d_i(u^{-1})$ is an elementary row transvection. Multiplication on the left by $d_i(u)$ multiplies the i'th row by u, and multiplication on the right by $d_i(u^{-1})$ multiplies the i'th column by u^{-1}; hence taking $e_{jk}(r)$ to $d_i(u)e_{jk}(r)d_i(u^{-1})$ leaves the diagonal entries of $e_{jk}(r)$ intact while possibly multiplying the only off-diagonal entry by u or u^{-1}. In any case the resulting matrix is one of the elementary row transvections $e_{jk}(r)$, $e_{jk}(ur)$ or $e_{jk}(u^{-1}r)$, so $D_n(R)$ normalizes $E_n(R)$. □

Theorem 4.2. *$E_n(R)$ contains matrices corresponding to the row operations*

(ii′) *multiplying the i'th row by -1 and interchanging it with the j'th row for $i \neq j$; and*

(iii′) *multiplying rows i and j for $i \neq j$ with units u and u^{-1}, respectively.*

Furthermore, $ML_n(R) \subseteq GE_n(R)$, and $GE_n(R)$ is exactly the subgroup generated by the matrices corresponding to row operations in the Gauss–Jordan elimination process.

Proof: Define for $i \neq j$ and $u \in R^*$ matrices p'_{ij} and $d'_i(u)$ in $E_n(R)$ by

$$p'_{ij} = e_{ij}(1)e_{ji}(-1)e_{ij}(1), \text{ and}$$
$$d'_{ij}(u) = e_{ij}(u)e_{ji}(-u^{-1})e_{ij}(u)e_{ij}(-1)e_{ji}(1)e_{ij}(-1).$$

It is straightforward to verify that p'_{ij} and $d'_{ij}(u)$ correspond to the row operations (ii′) and (iii′), respectively. It follows that $p_{ij} = d_j(-1)p'_{ij} \in GE_n(R)$, so that $P_n(R) \subseteq GE_n(R)$ and thereby $ML_n(R) \subseteq GE_n(R)$. The matrices corresponding to Gauss–Jordan elimination are products of matrices from $E_n(R)$, $P_n(R)$ and $D_n(R)$, so this shows that any such product is contained in $GE_n(R)$. □

Theorem 4.2 shows that it is possible to get very far in the Gauss–Jordan elimination process using only elementary row transvections from $E_n(R)$. These are all determinant preserving, so the closest one can hope to get to reducing a matrix $A \in GL_n(R)$ with row operations from $E_n(R)$ is to bring it in the form $\operatorname{diag}(\det_R A, 1, \ldots, 1)$. The following proposition describes exactly when this is possible.

Proposition 4.3. *The following are equivalent for $n \in \mathbb{N}$.*

(i) *$GE_n(R) = GL_n(R)$.*

(ii) *Each $A \in GL_n(R)$ can be reduced to I_n by using the Gauss–Jordan row operations* (i)-(iii) *from above.*

(iii) *Each $A \in GL_n(R)$ is t-row equivalent to a matrix $d_1(u) = \operatorname{diag}(u, 1, \ldots, 1)$, where $u = \det_R A$.*

Proof: "(i) $\Rightarrow$ (ii)": The group $GE_n(R)$ is generated by Gauss–Jordan row operations, so the assumption that $GE_n(R) = GL_n(R)$ means that any matrix $A \in GL_n(R)$ can be row operated into I_n and vice versa.

"(ii) $\Rightarrow$ (iii)": Given $A \in GL_n(R)$, the assumption together with Theorem 4.2 allows us to find matrices $D \in D_n(R)$ and $E \in E_n(R)$ such that $I_n = DEA$. Here D is in the form $D = \operatorname{diag}(u_1, \dots, u_n)$ for $u_1, \dots, u_n \in R^*$. Recalling the row operation (iii′) from Theorem 4.2 described by the matrices $d'_{ij}(u) \in E_n(R)$ for u a unit, we see that

$$D = \operatorname{diag}(u_1 \cdots u_n, 1, \dots, 1) d'_{21}(u_2) \cdots d'_{n1}(u_n).$$

Letting $u = u_1^{-1} \cdots u_n^{-1}$ and $E' = d'_{21}(u_2) \cdots d'_{n1}(u_n)E$, we now find that

$$\operatorname{diag}(u, 1 \dots, 1) = E'A.$$

This means that A is t-row equivalent to $\operatorname{diag}(u, 1, \dots, 1)$, and since E' is a product of matrices from $E_n(R)$, all of which have determinant 1, we must have $u = \det_R(\operatorname{diag}(u, 1, \dots, 1)) = \det_R(E'A) = \det_R A$.

"(iii) $\Rightarrow$ (i)": Given $A \in GL_n(R)$, the assumption provides the existence of a matrix $E \in E_n(R)$ and a unit $u \in R^*$ such that $A = E \operatorname{diag}(u, 1, \dots, 1)$. Since $E_n(R)D_n(R) = D_n(R)E_n(R) = GE_n(R)$, this shows that $A \in GE_n(R)$ as desired. □

Definition 4.4. The ring R is said to be *generalized Euclidean* if it satisfies any of the equivalent properties from Proposition 4.3.

Proposition 4.5. *If R is semilocal, then R is generalized Euclidean.*

Proof: Let $A = (a_{ij}) \in GL_n(R)$. We show by induction on n that A can be reduced to I_n using Gauss–Jordan row operations. The case $n = 1$ is clear, so suppose that $n > 1$ and that the statement holds for all smaller invertible matrices.

We commence by showing that, using row operations, we can obtain a unit in the $(1,1)$-entry of A. Let $\mathfrak{M}$ denote the (finite) set of maximal ideals containing a_{11}, and suppose that we have performed row operations so that the cardinality of $\mathfrak{M}$ is minimal, in the sense that no sequence of row operations can lead to a matrix whose $(1,1)$-entry is contained in fewer maximal ideals than the number of maximal ideals in $\mathfrak{M}$. We want to show that $\mathfrak{M}$ is empty, so that a_{11} must be a unit. Suppose therefore that $\mathfrak{M}$ is nonempty, and let $\mathfrak{m}_0 \in \mathfrak{M}$. Now, the entries in the first column of A cannot all be contained in the same maximal ideal, because then the determinant of A would also be contained in that maximal ideal and hence not be a unit. Thus we can find i so that $a_{i1} \notin \mathfrak{m}_0$. Using prime avoidance (Lemma 0.1), we can, for each maximal ideal $\mathfrak{m}$, find an element $x_{\mathfrak{m}}$ satisfying the condition that $x_{\mathfrak{m}}$ is contained in $\mathfrak{m}$ and in no other maximal ideal. We now perform a row operation on A, adding $\prod_{\mathfrak{m}\notin\mathfrak{M}} x_{\mathfrak{m}}$ times the i'th row to the first row. As its $(1,1)$-entry, the obtained matrix will have the element

$$a_{11} + a_{i1} \prod_{\mathfrak{m}\notin\mathfrak{M}} x_{\mathfrak{m}}.$$

This element cannot be contained in any of the maximal ideals from outside $\mathfrak{M}$, since this would imply that a_{11} was contained in such a maximal ideal. Furthermore, the element cannot be contained in the maximal ideal $\mathfrak{m}_0$, since this would imply that one of the factors of $a_{i1} \prod_{\mathfrak{m}\notin\mathfrak{M}} x_{\mathfrak{m}}$ is contained in $\mathfrak{m}_0$. Thus, row operations produced a matrix whose $(1,1)$ entry is contained in fewer maximal ideals than the number of maximal ideals in $\mathfrak{M}$. This contradicts our assumption, so $\mathfrak{M}$ must be empty, and hence a_{11} is a unit.

With a unit in the $(1,1)$-entry, we can now perform row operations to obtain a matrix A' in the form

$$A' = \begin{pmatrix} 1 & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} \end{pmatrix}.$$

The determinant of this matrix is still a unit and equals the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by deleting the first row and first column. Thus, according to the induction hypothesis, we can perform row operations on the bottom $n-1$ rows of A', thereby obtaining a matrix that only differs from the unit matrix in the last $n-1$ entries of the first row. Such a matrix is easily reduced to the unit matrix using row operations, and we have proved the proposition. □

Recall that if G is a group and $x, y \in G$, the *commutator of x and y* is the element $[x,y] = xyx^{-1}y^{-1}$. The elements x and y commute if and only if $[x,y] = 1$. The subgroup of G generated by all commutators is called the *commutator subgroup of G* and is denoted by $[G,G]$. Since $[x,y]^{-1} = [y,x]$, it consists of all finite-length products of commutators in G. The quotient $G_{\text{ab}} = G/[G,G]$ is known as the *Abelianization* of G; it is the largest Abelian quotient of G, in the sense that, for any subgroup H of G, H contains $[G,G]$ if and only if $H \lhd G$ and G/H is Abelian.

The next theorem shows that $E_n(R)$ in some sense is "completely non-Abelian".

Theorem 4.6. *If $n \geq 3$, then $[E_n(R), E_n(R)] = E_n(R)$.*

Proof: If $i, j \in \{1, \ldots, n\}$, we can find $k \in \{1, \ldots, n\}$ different from i and j. A simple calculation then shows that, for any $r \in R$, $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$. This proves the theorem. □

From Theorem 4.6 it follows that $E_n(R) \subseteq [GL_n(R), GL_n(R)]$: that is, the group of elementary matrices forms a subset of the commutator group of $GL_n(R)$.

We would like the group of elementary matrices to be exactly the commutator subgroup of $GL_n(R)$, but as we shall see below, this requires a little more "elbow room".

Definition 4.7. The *infinite-dimensional general linear group* is the multiplicative group $GL(R)$ of $\mathbb{N} \times \mathbb{N}$ matrices that are obtained from the matrix

$$I_\infty = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

by replacing an upper left corner I_n by an invertible matrix $A = (a_{ij}) \in GL_n(R)$ so that one gets the $\mathbb{N} \times \mathbb{N}$ matrix

$$A \oplus I_\infty = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots \\ \vdots & & \vdots & \vdots & \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots \\ 0 & \cdots & 0 & 1 & \cdots \\ \vdots & & \vdots & \vdots & \ddots \end{pmatrix}.$$

The composition in $GL(R)$ is given as follows. If $A \in GL_m(R)$ and $B \in GL_n(R)$,

$$A \oplus I_n = \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad B \oplus I_m = \begin{pmatrix} B & 0 \\ 0 & I_m \end{pmatrix}$$

are both in $GL_{m+n}(R)$, and we set $(A \oplus I_\infty)(B \oplus I_\infty) = ((A \oplus I_n)(B \oplus I_m)) \oplus I_\infty$. The neutral element of this composition is I_∞, and inverse elements are given by $(A \oplus I_\infty)^{-1} = A^{-1} \oplus I_\infty$ for $A \in GL_n(R)$. By identifying a matrix A in $GL_n(R)$ with the matrix $A \oplus I_\infty$ in $GL(R)$, $GL_n(R)$ can be identified with the subgroup $\{A \oplus I_\infty \,|\, A \in GL_n(R)\}$ of $GL(R)$. With these identifications we find that

$$R^* = GL_1(R) \subseteq GL_2(R) \subseteq \cdots$$

and $GL(R) = \bigcup_{n=1}^\infty GL_n(R)$. The group of *elementary matrices* is the subgroup $E(R)$ of $GL(R)$ generated by the elementary transvections

$$e_{ij}(r) = I_\infty + r\epsilon_{ij}$$

for $r \in R$ and $i \neq j$: that is, the $\mathbb{N} \times \mathbb{N}$ matrices obtained from I_∞ by replacing an off-diagonal entry with r. Under the identification of $A \in GL_n(R)$ with $A \oplus I_\infty \in GL(R)$, the usual elementary transvections $e_{ij}(r) \in E_n(R)$ become the $e_{ij}(r) \in GL(R)$, and hence

$$\{1\} = E_1(R) \subseteq E_2(R) \subseteq \cdots$$

and $E(R) = \bigcup_{n=1}^\infty E_n(R)$.

Lemma 4.8 (Whitehead). *If $n \geq 3$, then*

$$E_n(R) \subseteq [GL_n(R), GL_n(R)] \subseteq E_{2n}(R),$$

and it follows that $[GL(R), GL(R)] = E(R)$.

Proof: The first inclusion follows from Theorem 4.6. For the second inclusion, let $A, B \in GL_n(R)$. We want to show that $\left(\begin{smallmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I_n \end{smallmatrix}\right)$ is in $E_{2n}(R)$. Since

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix},$$

it suffices to show that a matrix in the form $\left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1} \end{smallmatrix}\right)$ for $A \in GL_n(R)$ is in $E_{2n}(R)$. Now, calculation shows that

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -A^{-1} & I_n \end{pmatrix} \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix},$$

and hence it suffices to show that matrices in the forms $\left(\begin{smallmatrix} I_n & A \\ 0 & I_n \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} I_n & 0 \\ A & I_n \end{smallmatrix}\right)$ for $A \in GL_n(R)$ are in $E_{2n}(R)$. This, however, follows immediately from two more calculations:

$$\begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} = I_{2n} + \sum_{1 \leq i,j \leq n} a_{ij} \epsilon_{i(n+j)} = \prod_{1 \leq i,j \leq n} (I_{2n} + a_{ij} \epsilon_{i(n+j)}) \quad \text{and}$$
$$\begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} = I_{2n} + \sum_{1 \leq i,j \leq n} a_{ij} \epsilon_{(n+i)j} = \prod_{1 \leq i,j \leq n} (I_{2n} + a_{ij} \epsilon_{(n+i)j}).$$

This shows that $E_n(R) \subseteq [GL_n(R), GL_n(R)] \subseteq E_{2n}(R)$. Now, if $A \in GL_m(R)$ and $B \in GL_n(R)$, then

$$[A \oplus I_\infty, B \oplus I_\infty] = [A \oplus I_n, B \oplus I_m] \oplus I_\infty,$$

and hence $[GL(R), GL(R)] = \bigcup_{n=1}^{\infty} [GL_n(R), GL_n(R)]$. From this follows that $[GL(R), GL(R)] = E(R)$. □

We are finally ready to present

Definition 4.9. The *first algebraic K-group* of R is the group

$$K_1(R) = GL(R)_{\text{ab}} = GL(R)/E(R).$$

Given $A \in GL_n(R)$, we denote the coset of $A \oplus I_\infty \in GL(R)$ in $K_1(R)$ by $[A]_R$ or, when there is no doubt about which ring is being referred to, simply by $[A]$.

For any $A \in GL_n(R)$ and any $m \in \mathbb{N}$, $[A] = [A \oplus I_m]$ in $K_1(R)$. We also have $[A] = [I_m \oplus A]$. To see this, note first that $[A] = [I_n \oplus A]$, since

$$I_n \oplus A = (A^{-1} \oplus A)(A \oplus I_n)$$

where $A^{-1} \oplus A$ is trivial in $K_1(R)$, since it belongs to $E_{2n}(R)$ as seen in the proof of Whitehead's lemma (Lemma 4.8). It follows that

$$[A] = [A \oplus I_{2m}] = [I_{2m+n} \oplus (A \oplus I_{2m})] = [I_{m+n} \oplus (I_m \oplus A)] = [I_m \oplus A].$$

A consequence of this is that, for matrices $A \in GL_m(R)$ and $B \in GL_n(R)$,

$$[AB] = [A][B] = [A \oplus I_n][I_m \oplus B] = [A \oplus B].$$

The usual determinant maps $\det_R \colon GL_n(R) \to R^*$ induce a determinant $\det_R \colon GL(R) \to R^*$ defined for $A \in GL_n(R)$ by

$$\det_R(A \oplus I_\infty) = \det_R A.$$

This is clearly well defined in the sense that the determinant of $A \oplus I_\infty$ is independent of choice of n and A. Since R^* is commutative, the map $\det_R \colon GL(R) \to R^*$ factors through the Abelianization of $GL(R)$, and hence there is an induced determinant map $K_1(R) \to R^*$, which we also denote $\det_R$, given for $A \in GL_n(R)$ by $\det_R [A] = \det_R A$.

The kernel of the usual determinant $\det_R \colon GL_n(R) \to R^*$ is denoted by $SL_n(R)$; it is the *special linear group* consisting of $n \times n$ matrices over R with determinant 1. The kernel of $\det_R \colon GL(R) \to R^*$ is denoted by $SL(R)$, and the kernel of $\det_R \colon K_1(R) \to R^*$ is denoted by $SK_1(R)$. Notice that since $K_1(R) = GL(R)/E(R)$ where $E(R) \subseteq SL(R)$, we must have $SK_1(R) = SL(R)/E(R)$.

The determinant map enables us in certain cases to compute $K_1(R)$ quite easily. The following proposition is the crucial step.

Proposition 4.10. *There is an isomorphism $K_1(R) \cong SK_1(R) \oplus R^*$. Whenever $SK_1(R)$ is trivial, this isomorphism is given by* $\det_R \colon K_1(R) \to R^*$.

Proof: Let $h \colon R^* \to K_1(R)$ be the homomorphism taking a unit $u \in R^*$ to the element $[u] \in K_1(R)$, the coset of the diagonal matrix $u \oplus I_\infty$. It is clear that $\det_R(h(u)) = \det_R([u]) = u$, so $\det_R \colon K_1(R) \to R^*$ is surjective and the following sequence is split exact.

$$\{1\} \longrightarrow SK_1(R) \longrightarrow K_1(R) \underset{h}{\overset{\det_R}{\rightleftarrows}} R^* \longrightarrow \{1\}.$$

It follows that $K_1(R) \cong SK_1(R) \oplus R^*$ and that $\det_R \colon K_1(R) \to R^*$ is an isomorphism whenever $SK_1(R)$ is trivial. □

We now realize the importance of Definition 4.4.

Theorem 4.11. *If R is generalized Euclidean, then* $\det_R: K_1(R) \to R^*$ *is an isomorphism.*

Proof: We need to show that $SK_1(R) = SL(R)/E(R)$ is trivial, but according to Proposition 4.3, any $n \times n$ matrix with determinant 1 is t-row equivalent to I_n. Thus, $SL(R) = E(R)$. □

We have previously seen that a ring homomorphism $\rho: R \to R'$ between (nontrivial, unitary and commutative) rings induces a group homomorphism $K_0(\rho): K_0(R) \to K_0(R')$ between the corresponding K_0-groups. We shall now see that the same property is satisfied when constructing K_1-groups. So suppose R and R' are (nontrivial, unitary and commutative) rings and that $\rho: R \to R'$ is a ring homomorphism. We claim that entry-wise application of ρ defines a group homomorphism $GL(\rho): GL(R) \to GL(R')$. To obtain this, note that application of ρ commutes with determinant: that is, if $A = (a_{ij})$ is an $n \times n$ matrix over R, then $A' = (\rho(a_{ij}))$ is an $n \times n$ matrix over R' whose determinant is $\det_{R'} A' = \rho(\det_R A)$. It follows that entry-wise application of ρ takes invertible matrices to invertible matrices, so entry-wise application of ρ defines a group homomorphism $GL(\rho): GL(R) \to GL(R')$. When Abelianizing domain as well as co-domain, $GL(\rho)$ collapses to a group homomorphism $K_1(R) \to K_1(R')$.

Definition 4.12. If R and R' are (nontrivial, unitary and commutative) rings, and $\rho: R \to R'$ is a ring homomorphism, then we denote by $K_1(\rho)$ the group homomorphism $K_1(R) \to K_1(R')$ defined by entry-wise application of ρ.

4.2 The localization sequence

Throughout this section, S denotes a (single) multiplicative system such that $S \cap \operatorname{Zd} R = \emptyset$, and R is assumed to be Noetherian.

The first algebraic K-groups are connected to the Grothendieck groups in an exact sequence known as the *localization sequence*. In this section we shall establish this fact and see how it allows some very nice reductions in the computation of some of the Grothendieck groups of the preceding chapter.

Let ρ_S denote the ring homomorphism $R \to S^{-1}R$ taking an element $x \in R$ to $\rho_S(x) = x/1$. Note that since S contains no zerodivisors, ρ_S must be injective. Note also that the units of $S^{-1}R$ are those x/s such that $y \in R$ and $t \in S$ exist with $xy = st$; in particular, R^* is mapped into $(S^{-1}R)^*$ under ρ_S, and the denominators as well as the numerators of the fractions in $(S^{-1}R)^*$ are nonzerodivisors of R.

Recall from Definitions 2.18 and 4.12 that there are induced homomorphisms $K_0(\rho_S): K_0(R) \to K_0(S^{-1}R)$ and $K_1(\rho_S): K_1(R) \to K_1(S^{-1}R)$. We shall connect $K_1(R) \to K_1(S^{-1}R)$ to $K_0(R) \to K_0(S^{-1})$ via the group $G_0^R(\text{f,pd,}S\text{-tor})$ in an exact sequence

$$K_1(R) \xrightarrow{K_1(\rho_S)} K_1(S^{-1}R) \xrightarrow{\delta} G_0^R(\text{f,pd},S\text{-tor}) \xrightarrow{\chi} K_0(R) \xrightarrow{K_0(\rho_S)} K_0(S^{-1}R).$$

Here χ is the Euler characteristic from Theorem 2.13. The map δ still needs to be defined and the fact that the above sequence is exact still needs to be shown.

Suppose that we are given an element $[A]$ of $K_1(S^{-1}R)$, where A is an invertible matrix in $GL_n(S^{-1}R)$. If the entries of A are in R (that is, have 1 as a common divisor), then the obvious choice for $\delta([A])$ is simply to let $\delta([A]) = [R^n/AR^n]$. This indeed defines an element of $G_0^R(\text{f,pd},S\text{-tor})$: the module R^n/AR^n is finitely generated by n elements, has a projective resolution

$$0 \longrightarrow R^n \xrightarrow{A} R^n \longrightarrow R^n/AR^n \longrightarrow 0$$

(A is injective since $\det_R A$ is a unit in $S^{-1}R$ and thereby a non-zerodivisor in R), and is S-torsion since A becomes invertible when localized at S.

Now, in case A does not have entries in R, we can always find a common divisor $s \in S$ such that (sA), which is still a member of $GL_n(S^{-1}R)$, has entries in R. Since $[A] = [(sA)(s^{-1}I_n)] = [sA][s]^{-n}$ in $K_1(S^{-1}R)$, we must then have $\delta([A]) = \delta([sA]) - n\delta([s]) = [R^n/(sA)R^n] - n[R/sR]$. Thus, the requirement that δ should map elements $[A]$, where A is a matrix with entries in R, to $[R^n/AR^n]$ completely determines δ. It still remains to prove that δ exists.

Theorem 4.13. *There is a homomorphism $\delta\colon K_1(S^{-1}R) \to G_0^R(\text{f,pd},S\text{-tor})$ given for $A \in GL_n(S^{-1}R)$ by $\delta([A]) = [R^n/(sA)R^n] - n[R/sR]$, where $s \in S$ is chosen so that (sA) has entries in R.*

Proof: Let us first try to construct a homomorphism

$$d_n\colon GL_n(S^{-1}R) \to G_0^R(\text{f,pd},S\text{-tor})$$

by setting $d_n(A) = [R^n/(sA)R^n] - n[R/sR]$, where $s \in S$ is chosen so that (sA) has entries in R. We already know that $[R^n/(sA)R^n] - n[R/sR]$ is a well-defined element of $G_0^R(\text{f,pd},S\text{-tor})$, so it only remains to verify that this element is independent of the choice of s and that $d_n(AB) = d_n(A) + d_n(B)$ whenever $A, B \in GL_n(S^{-1}R)$. For the first part, suppose that $t \in S$ is another element such that (tA) has entries in R. Since s and t are non-zerodivisors, and (sA) and (tA) are injective over R, we have exact sequences

$$0 \longrightarrow R^n/tR^n \xrightarrow{(sA)} R^n/(stA)R^n \longrightarrow R^n/(sA)R^n \longrightarrow 0$$

and

$$0 \longrightarrow R/tR \xrightarrow{s} R/stR \longrightarrow R/sR \longrightarrow 0$$

in $\mathcal{C}_0^R$(f,pd,S-tor), proving that in G_0^R(f,pd,S-tor),

$$\begin{aligned}[R^n/(sA)R^n] - n[R/sR] &= [R^n/(stA)R^n] - [R^n/tR^n] - n([R/stR] - [R/tR]) \\ &= [R^n/(stA)R^n] - n[R/stR].\end{aligned}$$

Interchanging s and t, we similarly get

$$[R^n/(tA)R^n] - n[R/tR] = [R^n/(stA)R^n] - n[R/stR],$$

proving that $d_n(A)$ is independent of the choice of s. Thus, d_n is well defined as a function $GL_n(S^{-1}R) \to G_0^R$(f,pd,S-tor).

Now, if $A, B \in GL_n(S^{-1}R)$, we choose $s \in S$ so that (sA) and (sB) both have entries in R. The exact sequences

$$0 \longrightarrow R^n/(sB)R^n \xrightarrow{(sA)} R^n/(s^2AB)R^n \longrightarrow R^n/(sA)R^n \longrightarrow 0$$

and

$$0 \longrightarrow R/sR \xrightarrow{s} R/s^2R \longrightarrow R/sR \longrightarrow 0$$

in $\mathcal{C}_0^R$(f,pd,S-tor) then show that in G_0^R(f,pd,S-tor),

$$\begin{aligned}d_n(AB) &= [R^n/(s^2AB)R^n] - n[R/s^2R] \\ &= [R^n/(sA)R^n] + [R^n/(sB)R^n] - 2n[R/sR] \\ &= d_n(A) + d_n(B).\end{aligned}$$

This establishes d_n as a homomorphism $GL_n(S^{-1}R) \to G_0^R$(f,pd,S-tor).

Now, given $m \in \mathbb{N}$, projections onto the first n and the last m coordinates determine isomorphisms

$$R^{n+m}/s(A \oplus I_m)R^{n+m} \cong R^n/(sA)R^n \oplus R^m/sR^m$$

and

$$R^{n+m}/sR^{n+m} \cong R^n/sR^n \oplus R^m/sR^m,$$

proving that $d_{n+m}(A \oplus I_m) = d_n(A)$. Consequently, the maps d_n extend to a homomorphism $d\colon GL(S^{-1}R) \to G_0^R$(f,pd,$S$-tor) given by $d(A) = d_n(A)$ for each $A \in GL_n(S^{-1}R)$. Since G_0^R(f,pd,S-tor) is Abelian, d induces a homomorphism $\delta\colon K_1(S^{-1}R) \to G_0^R$(f,pd,$S$-tor) given by $\delta([A]) = d_n(A)$ for $A \in GL_n(S^{-1}R)$. □

Now that all the maps in the localization sequence have been introduced, we are ready to prove that it is exact. The key to this will be Corollary 3.40, which enables us to switch between the groups G_0^R(f,pd,S-tor), G_1^R(f,P|S-tor) and $G_\square^R$(f,P|S-tor), using the isomorphisms $\mathcal{H}$, $\mathcal{R}$ and ι. Switching from G_0^R(f,pd,S-tor) to G_1^R(f,P|S-tor), the homomorphism δ is replaced by $\iota^{-1} \circ \mathcal{R} \circ \delta$, which takes an element $[A]$ in $K_1(S^{-1}R)$ to the element

$$[0 \longrightarrow R^n \xrightarrow{(sA)} R^n \longrightarrow 0] - n[0 \longrightarrow R \xrightarrow{s} R \longrightarrow 0]$$

in $G_1^R(\text{f,P}|S\text{-tor})$. Meanwhile, the homomorphism χ is replaced by $\chi \circ \mathcal{H}$, which is equal to the homomorphism $\mathcal{A}$ from Theorem 2.10, taking an element $[X]$ in $G_1^R(\text{f,P}|S\text{-tor})$ to the element $[X_0] - [X_1]$ in $K_0(R)$. We shall denote these maps by δ and χ as well, leaving it to the reader to determine which of the groups $G_0^R(\text{f,pd},S\text{-tor})$ and $G_1^R(\text{f,P}|S\text{-tor})$ is being referred to.

First, however, we need a few lemmas.

Lemma 4.14. *Given a complex Y in $\mathcal{C}_1^R(\text{f,P}|S\text{-tor})$, there exists a complex $\tilde{Y}$ in $\mathcal{C}_1^R(\text{f,P}|S\text{-tor})$ such that $Y \oplus \tilde{Y}$ is a complex on the form*

$$0 \longrightarrow R^n \xrightarrow{A} R^n \longrightarrow 0$$

for some $n \in \mathbb{N}$ and some matrix $A \in GL_n(S^{-1}R)$ with entries in R. In particular, every element of $G_1^R(\text{f,P}|S\text{-tor})$ is in the form $[X]$ modulo $\operatorname{im}\delta$ for some $X \in \mathcal{C}_1^R(\text{f,P}|S\text{-tor})$.

Proof: Let Y be a given complex as above, and recall from Proposition 3.2 that Y has an S-contraction: that is, a homomorphism $\beta_0 \colon Y_0 \to Y_1$ such that $\partial_1^Y \beta_0 = s\mathbb{1}_{Y_0}$ and $\beta_0 \partial_1^Y = s\mathbb{1}_{Y_1}$ for some fixed $s \in S$. When localized at S, ∂_1^Y and $\partial_1^Y \beta_0 = s\mathbb{1}_{Y_0}$ become isomorphisms, and hence so does β_0, and it follows that the complex

$$\bar{Y} = \quad 0 \longrightarrow Y_0 \xrightarrow{\beta_0} Y_1 \longrightarrow 0$$

concentrated in degrees 1 and 0 must be homologically S-torsion: that is, $\bar{Y}$ is a complex in $\mathcal{C}_1^R(\text{f,P}|S\text{-tor})$.

Consider now the complex $Y \oplus \bar{Y}$, which has the module $Y_0 \oplus Y_1$ in both of its nontrivial degrees. The module $Y_0 \oplus Y_1$ is finitely generated and projective, so we can find a finitely generated and projective module L such that $Y_0 \oplus Y_1 \oplus L \cong R^n$ for some $n \in \mathbb{N}_0$. Letting Z denote the exact complex $0 \to L \to L \to 0$ concentrated in degrees 1 and 0, the direct sum $Y \oplus \bar{Y} \oplus Z$ is now nothing but a complex

$$0 \longrightarrow R^n \xrightarrow{A} R^n \longrightarrow 0,$$

which is in $G_1^R(\text{f,P}|S\text{-tor})$. Since the complex becomes exact when localized at S, the corresponding matrix A must be in $GL_n(S^{-1}R)$. Denoting by $\tilde{Y}$ the direct sum $\bar{Y} \oplus Z$, this proves the first part of the lemma.

From previous observations, we know that every element of $G_1^R(\text{f,P}|S\text{-tor})$ can be written in the form $[X] - [Y]$ for $X, Y \in G_1^R(\text{f,P}|S\text{-tor})$. It now follows that $[Y] + [\tilde{Y}] = \delta(A)$, and hence $[X] - [Y]$ modulo $\operatorname{im}\delta$ equals $[X] + [\tilde{Y}] = [X \oplus \tilde{Y}]$. This proves the last part of the lemma. □

Lemma 4.15. *If $A, B \in M_n(R)$ are injective $n \times n$ matrices and $\phi \colon R^n/AR^n \to R^n/BR^n$ is an isomorphism, then invertible matrices $C, D \in GL_{2n}(R)$ exist, such* *that there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^n \oplus R^n & \xrightarrow{\left(\begin{smallmatrix} A & 0 \\ 0 & I_n \end{smallmatrix}\right)} & R^n \oplus R^n & \xrightarrow{(\pi_A\ 0)} & R^n/AR^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & R^n \oplus R^n & \xrightarrow{\left(\begin{smallmatrix} B & 0 \\ 0 & I_n \end{smallmatrix}\right)} & R^n \oplus R^n & \xrightarrow{(\pi_B\ 0)} & R^n/BR^n & \longrightarrow & 0 \end{array}$$

in which π_A and π_B denote projection maps.

PROOF: The rows of the diagram are clearly exact, so we just need to find the matrices C and D. We already know that ϕ lifts to a map $\alpha\colon R^n \to R^n$, and similarly that ϕ^{-1} lifts to a map $\beta\colon R^n \to R^n$ (see, for example, [HS97, Theorem IV.4.1]). Thus, $\alpha, \beta \in M_n(R)$ are matrices such that $\pi_B\alpha = \phi\pi_A$ and $\pi_A\beta = \phi^{-1}\pi_B$. Now let

$$D = \begin{pmatrix} 2\alpha - \alpha\beta\alpha & \alpha\beta - I_n \\ I_n - \beta\alpha & \beta \end{pmatrix},$$

which is equal to the product

$$\begin{pmatrix} I_n & \alpha \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -\beta & I_n \end{pmatrix} \begin{pmatrix} I_n & \alpha \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix}$$

and therefore is invertible with determinant 1. (Note that D corresponds to the matrix $d'_{12}(u)$ from Theorem 4.2 in which the unit u has been replaced by α and its inverse u^{-1} by β.) The equations $\pi_B\alpha = \phi\pi_A$ and $\pi_A\beta = \phi^{-1}\pi_B$ show that our choice of D makes the right square of the diagram above commute. It follows, in particular, that the kernels of $\begin{pmatrix}\pi_A & 0\end{pmatrix}$ and $\begin{pmatrix}\pi_B & 0\end{pmatrix}$ are isomorphic under D: that is, an isomorphism $\psi\colon AR^n \oplus R^n \to BR^n \oplus R^n$ induced by D exists. But the domain as well as the co-domain of ψ are isomorphic to R^{2n} by the maps $\left(\begin{smallmatrix} A & 0 \\ 0 & I_n \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} B & 0 \\ 0 & I_n \end{smallmatrix}\right)$, and hence ψ can be translated into an isomorphism $C\colon R^{2n} \to R^{2n}$, making the diagram

$$\begin{array}{ccc} R^n \oplus R^n & \xrightarrow{\left(\begin{smallmatrix} A & 0 \\ 0 & I_n \end{smallmatrix}\right)} & AR^n \oplus R^n \\ \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle \psi} \\ R^n \oplus R^n & \xrightarrow{\left(\begin{smallmatrix} B & 0 \\ 0 & I_n \end{smallmatrix}\right)} & BR^n \oplus R^n \end{array}$$

commutative. Consequently, $C \in GL_{2n}(R)$ is an invertible matrix, making the left square of the previous diagram commutative, and the lemma has been proven. □

Theorem 4.16. *The sequence*

$$K_1(R) \xrightarrow{K_1(\rho_S)} K_1(S^{-1}R) \xrightarrow{\delta} G_0^R(\text{f,pd},S\text{-tor}) \xrightarrow{\chi} K_0(R) \xrightarrow{K_0(\rho_S)} K_0(S^{-1}R)$$

is exact.

Proof: *Exactness at $K_1(S^{-1}R)$*: The composition $\delta \circ K_1(\rho_S)$ takes $[A]_R \in K_1(R)$ to $[R^n/AR^n]$, where R^n/AR^n is the zero module, since A is invertible. Consequently, $\delta \circ K_1(\rho_S) = 0$.

Conversely, suppose that $[A]_{S^{-1}R}$ is in the kernel of δ: that is, $[R^n/(sA)R^n] = [R^n/sR^n]$ for $s \in S$ chosen so that (sA) has entries in R. We want to show that $[A]_{S^{-1}R}$ is in the image of $K_1(\rho_S)$. Let us show even more generally that, if A and B are matrices in $GL_n(S^{-1}R)$ with entries in R such that $[R^n/AR^n] = [R^n/BR^n]$, then $[AB^{-1}]_{S^{-1}R}$ is in the image of $K_1(\rho_S)$.

Suppose first that the equation $[R^n/AR^n] = [R^n/BR^n]$ is caused by an isomorphism $\phi\colon R^n/AR^n \to R^n/BR^n$. According to Lemma 4.15 invertible matrices $C, D \in GL_{2n}(R)$ then exist such that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^n \oplus R^n & \xrightarrow{\left(\begin{smallmatrix} A & 0 \\ 0 & I_n \end{smallmatrix}\right)} & R^n \oplus R^n & \xrightarrow{(\pi_A\ 0)} & R^n/AR^n & \longrightarrow & 0 \\
 & & \downarrow C & & \downarrow D & & \downarrow \phi & & \\
0 & \longrightarrow & R^n \oplus R^n & \xrightarrow{\left(\begin{smallmatrix} B & 0 \\ 0 & I_n \end{smallmatrix}\right)} & R^n \oplus R^n & \xrightarrow{(\pi_B\ 0)} & R^n/BR^n & \longrightarrow & 0
\end{array}$$

in which π_A and π_B denote projection maps. It follows that, in $K_1(S^{-1}R)$,

$$[AB^{-1}] = [A \oplus I_n][B \oplus I_n]^{-1} = [D]^{-1}[B \oplus I_n][C][B \oplus I_n]^{-1} = [CD^{-1}]$$

is in the image of $K_1(\rho_S)$.

In the general case, we cannot be sure that the equation $[R^n/AR^n] = [R^n/BR^n]$ derives from an isomorphism as above. However, Proposition 2.6 ensures the existence of short exact sequences $0 \to L \to M_i \to N \to 0$, $i = 1, 2$, in $\mathcal{C}_0^R(\text{f,pd},S\text{-tor})$ such that $(R^n/AR^n) \oplus M_1 \cong (R^n/BR^n) \oplus M_2$. By switching temporarily to $\mathcal{C}_1^R(\text{f,P}|S\text{-tor})$, we can even make sure that the modules L, M_1, M_2 and N have projective dimensions at most 1. Now apply Lemma 4.14 to find modules L' and N' such that $L \oplus L'$ and $N \oplus N'$ are co-kernels of square matrices with entries in R. Adding an identity matrix to either of these matrices, we can even find that the matrices have the same dimension $m \in \mathbb{N}$. By adding the short exact sequences $0 \to L' \to L' \to 0 \to 0$ and $0 \to 0 \to N' \to N' \to 0$ to each of the short exact sequences $0 \to L \to M_i \to N \to 0$ and redefining M_i to be the module $M_i \oplus L' \oplus N'$ for $i = 1, 2$, we obtain short exact sequences in the form

$$0 \to R^m/ER^m \to M_i \to R^m/FR^m \to 0,$$

where $E, F \in GL_m(S^{-1}R)$ are matrices with entries in R. We still have that $R^n/AR^n \oplus M_1 \cong R^n/BR^n \oplus M_2$. From the obvious projective resolutions of R^m/ER^m and R^m/FR^m, we can construct a projective resolution of M_i whose modules are the direct sum of the corresponding modules in the resolutions of R^m/ER^m and R^m/FR^m. It follows that the module M_i is also the co-kernel of a matrix $G_i \in GL_{2n}(S^{-1}R)$ with entries in R, so we now have

$$R^n/AR^n \oplus R^{2m}/G_1R^{2m} \cong R^n/BR^n \oplus R^{2m}/G_2R^{2m}.$$

This is the situation from before, so we can immediately conclude that, in $K_1(S^{-1}R)$,

$$[A][G_1][B]^{-1}[G_2]^{-1} = [A \oplus G_1][B \oplus G_2]^{-1}$$

is in the image of $K_1(\rho_S)$. To derive that $[A][B]^{-1}$ is in the image of $K_1(\rho_S)$, it therefore only remains to prove that $[G_1] = [G_2]$ in $K_1(S^{-1}R)$.

By the construction of G_i there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^m & \xrightarrow{\left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right)} & R^{2m} & \xrightarrow{\left(\begin{smallmatrix} 0 & I_m \end{smallmatrix}\right)} & R^m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle G_i} & & \downarrow{\scriptstyle F} & & \\
0 & \longrightarrow & R^m & \xrightarrow{\left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right)} & R^{2m} & \xrightarrow{\left(\begin{smallmatrix} 0 & I_m \end{smallmatrix}\right)} & R^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^m/ER^m & \longrightarrow & R^{2m}/G_iR^{2m} & \longrightarrow & R^m/FR^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From the commutativity of the diagram, it follows that G_i must be in the form

$$G_i = \begin{pmatrix} E & G_i' \\ 0 & F \end{pmatrix}$$

for some matrix $G_i' \in M_m(R)$. The proof of Whitehead's lemma (Lemma 4.8) showed that any matrix in the form $\left(\begin{smallmatrix} I_m & \star \\ 0 & I_m \end{smallmatrix}\right)$ is trivial in $K_1(S^{-1}R)$, so from the calculation

$$G_i = \begin{pmatrix} E & G_i' \\ 0 & F \end{pmatrix} = \begin{pmatrix} I_m & G_i'F^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}$$

it follows that $[G_i] = [E \oplus F]$ in $K_1(S^{-1}R)$. In particular, $[G_1] = [G_2]$ as desired.

Exactness at G_0^R(f,pd,S-tor): Suppose $A \in GL_n(S^{-1}R)$ such that $[A]$ is an arbitrary element of $K_1(S^{-1}R)$. Then $\delta([A]) = [R^n/(sA)R^n] - n[R/sR]$ for $s \in S$ chosen so that (sA) has entries in R, and since we have projective resolutions

$$0 \longrightarrow R^n \xrightarrow{(sA)} R^n \longrightarrow R^n/(sA)R^n \longrightarrow 0$$

and

$$0 \longrightarrow R \xrightarrow{s} R \longrightarrow R/sR \longrightarrow 0,$$

it follows that $\chi \circ \delta([A]) = [R^n] - [R^n] + [R] - [R] = 0$.

To show conversely that the kernel of χ is contained in the image of δ, we switch from G_0^R(f,pd,S-tor) to the isomorphic group G_1^R(f,P|S-tor). Using Lemma 4.14, it suffices to assume that we are given an element $[X] \in G_1^R$(f,P|S-tor) that is in the kernel of χ: that is, such that $[X_0] = [X_1]$ in $K_0(R)$. It then follows from Corollary 2.7 (together with [Mag02, Proposition 2.21]) that we can find a finitely generated projective module L such that $X_0 \oplus L \cong X_1 \oplus L \cong R^n$ for some $n \in \mathbb{N}$. Letting Z denote the exact complex $0 \to L \to L \to 0$ concentrated in degrees 1 and 0, $X \oplus Z$ is then nothing but a complex $0 \to R^n \to R^n \to 0$ that becomes exact when localizing at S. The matrix associated with this complex must therefore be in $GL_n(S^{-1}R)$, so $[X] = [X \oplus Z]$ is in $\operatorname{im}\delta$.

Exactness at $K_0(R)$: We immediately switch from $G_0^R(\text{f,pd,}S\text{-tor})$ to the isomorphic group $G_1^R(\text{f,P}|S\text{-tor})$. Suppose X is a complex in $\mathcal{C}_1^R(\text{f,P}|S\text{-tor})$. Then $S^{-1}X$ is exact, so $S^{-1}X_0 \cong S^{-1}X_1$, and it follows that $K_0(\rho_S) \circ \chi([X]) = [S^{-1}X_0] - [S^{-1}X_1] = 0$ in $K_0(S^{-1}R)$.

Conversely, suppose that $[M]-[N]$ is an element in the kernel of $K_0(\rho_S)$: that is, such that $[S^{-1}M] = [S^{-1}N]$ in $K_0(S^{-1}R)$. Using Corollary 2.7, we can then find $n \in \mathbb{N}_0$ such that $S^{-1}M \oplus (S^{-1}R)^n \cong S^{-1}N \oplus (S^{-1}R)^n$. It follows that there is a homomorphism $f\colon N \oplus R^n \to M \oplus R^n$ that induces an isomorphism

$$S^{-1}f\colon S^{-1}N \oplus (S^{-1}R)^n \to S^{-1}M \oplus (S^{-1}R)^n$$

when localized at S (cf. [Eis95, Proposition 2.10]), and hence

$$X = \quad 0 \longrightarrow N \oplus R^n \xrightarrow{f} M \oplus R^n \longrightarrow 0$$

is a complex in $\mathcal{C}_1^R(\text{f,P}|S\text{-tor})$ with $\chi([X]) = [M \oplus R^n] - [N \oplus R^n] = [M] - [N]$ as desired. □

Corollary 4.17. *If R is local, $K_0(\rho_S)$ is injective, and the sequence*

$$K_1(R) \xrightarrow{K_1(\rho_S)} K_1(S^{-1}R) \xrightarrow{\delta} G_0^R(\text{f,pd,}S\text{-tor}) \longrightarrow 0$$

is exact.

Proof: As shown in Example 2.14, there is an isomorphism $\mathbb{Z} \to K_0(R)$ given by $1 \mapsto [R]$ and an injection $\mathbb{Z} \to K_0(S^{-1}R)$ given by $1 \mapsto [S^{-1}R]$. Since $K_0(\rho_S)$ maps $[R]$ to $[S^{-1}R]$, it follows that $K_0(\rho_S)$ must be injective, and Theorem 4.16 implies the exactness of the sequence. □

Corollary 4.17 allows a determinant map

$$\det_S\colon G_0^R(\text{f,pd,}S\text{-tor}) \to (S^{-1}R)^*/R^*$$

to be induced from the determinant map $\det_{S^{-1}R}\colon K_1(S^{-1}R) \to (S^{-1}R)^*$, such that the diagram

$$\begin{array}{ccccccccc}
 & & K_1(R) & \xrightarrow{K_1(\rho_S)} & K_1(S^{-1}R) & \xrightarrow{\delta} & G_0^R(\text{f,pd,}S\text{-tor}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \det_R} & & \downarrow{\scriptstyle \det_{S^{-1}R}} & & \downarrow{\scriptstyle \det_S} & & \\
0 & \longrightarrow & R^* & \xrightarrow{\rho_S} & (S^{-1}R)^* & \longrightarrow & (S^{-1}R)^*/R^* & \longrightarrow & 0
\end{array}$$

is commutative: given $[M]$ in $G_0^R(\text{f,pd},S\text{-tor})$, we choose $[A]$ in $K_1(S^{-1}R)$ so that $\delta([A]) = [M]$, and we define $\det_S[M]$ to be the coset of $\det_{S^{-1}R}[A]$ in $(S^{-1}R)^*/R^*$. This is well defined, for if we choose $[A]$ and $[A']$ in $K_1(S^{-1}R)$ so that $\delta([A]) = \delta([A']) = [M]$, then $[A^{-1}A']$ is in the image of $K_1(\rho_S)$, and hence $(\det_{S^{-1}R}[A])^{-1}\det_{S^{-1}R}[A']$ is in the image of ρ_S.

Definition 4.18. When R is local, the homomorphism $G_0^R(\text{f,pd},S\text{-tor}) \to (S^{-1}R)^*$ induced by $\det_{S^{-1}R}\colon K_1(S^{-1}R) \to (S^{-1}R)^*$ is denoted by $\det_S$. Given a module M in $\mathcal{C}_0^R(\text{f,pd},S\text{-tor})$, we denote $\det_S[M]$ simply by $\det_S M$.

When R is local, the surjection δ shows that $G_0^R(\text{f,P},S\text{-tor})$ is generated by elements $[R^n/AR^n]$, where $n \in \mathbb{N}$ and A is a matrix in $GL_n(S^{-1}R)$ with entries in R, or equivalently, an $n \times n$ matrix over R that becomes invertible when localized at S. Given such an element $[R^n/AR^n]$, $\det_S[R^n/AR^n]$ is the coset of $\det_R A/1$ in $(S^{-1}R)^*/R^*$.

If $S^{-1}R$ is generalized Euclidean, everything looks much nicer:

Corollary 4.19. *If R is local and $S^{-1}R$ is generalized Euclidean, then there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(R) & \xrightarrow{K_1(\rho_S)} & K_1(S^{-1}R) & \xrightarrow{\delta} & G_0^R(\text{f,pd},S\text{-tor}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \det_R} & & \downarrow{\scriptstyle \det_{S^{-1}R}} & & \downarrow{\scriptstyle \det_S} & & \\
0 & \longrightarrow & R^* & \xrightarrow{\rho_S} & (S^{-1}R)^* & \longrightarrow & (S^{-1}R)^*/R^* & \longrightarrow & 0
\end{array}$$

in which the rows are exact and the columns are isomorphisms.

Proof: According to Proposition 4.5, both R and $S^{-1}R$ are generalized Euclidean, so it follows from Theorem 4.11 that the determinant maps $\det_R$ and $\det_{S^{-1}R}$ are isomorphisms. This immediately implies that $\det_S$ is an isomorphism as well and that $K_1(\rho_S)$ is injective. □

The fact that $\det_S$ is an isomorphism in the above setting shows that every element in $G_0^R(\text{f,pd},S\text{-tor})$ is in the form $[R/u] - [R/s]$, where $u \in R$ and $s \in S$ are elements such that $u/s \in (S^{-1}R)^*$. It turns out that, for any module M in $\mathcal{C}_0^R(\text{f,pd},S\text{-tor})$, $[M] = [R/u]$ for some $u \in R$ with $u/1 \in (S^{-1}R)^*$. The remainder of this chapter is dedicated to proving this.

In [Mac65] MacRae introduced an ideal, later known as the *MacRae ideal*, associated to every finitely generated module of finite projective dimension and grade greater than or equal to 1. Letting M be such a module, the MacRae ideal is denoted by $G_R(M)$. We will not present the actual definition of the MacRae ideal here, nor will we prove the theorem below, which describes some of the properties of the MacRae ideal in the case where R is Noetherian and local (cf. [Fox82b, Lemma (0.1)]).

Theorem 4.20. *Suppose that R is a Noetherian, local ring and that L, M and N are modules in $\mathcal{C}_0^R(\text{f,pd,gr} \geq 1)$. Then*

(i) *$G_R(M)$ is a principal ideal;*

(ii) *$G_R(M)$ is generated by $\det_R A$ whenever $M \cong R^n/AR^n$ for some injective matrix A;*

(iii) *$G_R(M) \neq R$ if and only if $\operatorname{grade}_R M = 1$;*

(iv) *$G_R(M) = G_R(L)G_R(N)$ whenever there is a short exact sequence $0 \to L \to M \to N \to 0$; and*

(v) *$V_R(G_R(M)) \subseteq \operatorname{Supp}_R M$.* □

The theorem below shows that the determinant $\det_S M$ from Definition 4.18 generates the MacRae ideal $G_R(M)$. For the proof of this, note that, if M is a module in $\mathcal{C}_0^R(\text{f,pd},S\text{-tor})$, then $\operatorname{grade}_R M \geq 1$ since S contains only non-zerodivisors, and hence the MacRae ideal is defined for M.

Theorem 4.21. *Suppose that R is Noetherian and local, and that M is a module in $\mathcal{C}_0^R(\text{f,pd},S\text{-tor})$. Then $\det_S M$ is the coset in $(S^{-1}R)^*/R^*$ of an element $u/1$ such that $G_R(M) = \langle u \rangle$.*

Proof: The proof is by induction on $p = \operatorname{pd}_R M$. The cases $p \leq 0$ are trivial, for in these cases M is projective and hence free, and the only way for M to be S-torsion is then if $M = 0$, in which case $\det_S M = [1/1]_{R^*}$ and $G_R(M) = R$ (the latter by Theorem 4.20(iii)).

In the case that $p = 1$, M is the homology of a complex

$$0 \longrightarrow R^n \xrightarrow{A} R^m \longrightarrow 0$$

for $m, n \in \mathbb{N}_0$ and an injective $m \times n$ matrix A. Now, the fact that M is S-torsion implies that $\operatorname{Ann}_R M \neq 0$, and from Theorem 0.8 it then follows that the (traditional) Euler characteristic of M is $\chi^R(M) = 0$. Since the Euler characteristic is additive on short exact sequences, the exact sequence $0 \to R^n \to R^m \to M \to 0$ shows that $m = n$. Thus, M is the co-kernel of a matrix A in $GL_n(S^{-1}R)$ with entries in R, and it follows that $\det_S M$ is the coset of $\det_R A/1$ in $(S^{-1}R)^*/R^*$. Conversely, Theorem 4.20(ii) implies that $G_R(M) = \langle \det_R A \rangle$.

For the general case, suppose that $p > 1$ and that the theorem has been proven for smaller values of p. Choose a finitely generated free module F and a surjective homomorphism $f\colon F \to M$. Since M is S-torsion, we can choose $s \in S \cap \operatorname{Ann}_R M$, and we obtain an induced homomorphism $\bar{f}\colon F/sF \to M$ that is also surjective. Letting K denote the kernel of $\bar{f}$, we then have an exact sequence

$$0 \to K \to F/sF \to M \to 0. \tag{4.1}$$

Since $\mathrm{pd}_R\, F/sF = 1$, Theorem 0.7 yields $\mathrm{pd}_R\, K = p-1$. By construction, F/sF and K are S-torsion, so F and K are modules in $\mathcal{C}_0^R(\mathrm{f,pd},S\text{-tor})$, and the induction hypothesis implies that $\det_S(F/sF) = [v/1]_{R^*}$ and $\det_S K = [w/1]_{R^*}$ for elements $v, w \in R$ such that $G_R(F/sF) = \langle v \rangle$ and $G_R(K) = \langle w \rangle$.

Let $u \in R$ be a generator for $G_R(M)$. From the short exact sequence in (4.1) together with Theorem 4.20(iv), we then find that $\langle v \rangle = \langle wu \rangle$ and that $[v/1]_{R^*} = [w/1]_{R^*} \det_S M$. From the first equation, it follows that $[v/1]_{R^*} = [wu/1]_{R^*}$, and from the second equation it then follows that $\det_S M = [v/1]_{R^*}[w/1]_{R^*}^{-1} = [u/1]_{R^*}$ as desired. □

Theorem 4.21 shows that $G_R(M)$ is generated by an element $u \in R$ satisfying the condition that $u/1$ is a unit in $S^{-1}R$. Consequently, there exists an element $u' \in R$ such that uu' is in S, and since this element also is in the annihilator of $R/G_R(M)$, it follows that $R/G_R(M)$ is a module in $\mathcal{C}_0^R(\mathrm{f,pd},S\text{-tor})$. Using Theorem 4.21 together with Corollary 4.19, we therefore immediately obtain the following corollary.

Corollary 4.22. *If R is Noetherian and local, $S^{-1}R$ is generalized Euclidean and M is a module in $\mathcal{C}_0^R(\mathrm{f,pd},S\text{-tor})$, then $[M] = [R/G_R(M)]$ in $G_0^R(\mathrm{f,pd},S\text{-tor})$.* □

The *grade conjecture* states that, if R is Noetherian and local and M is finitely generated with finite projective dimension, then $\mathrm{grade}_R\, M + \dim_R M = \dim R$. We have already noted in the preliminaries that Theorem 0.8 implies that the grade conjecture holds whenever $\mathrm{grade}_R\, M = 0$. This chapter concludes by showing that the properties of the MacRae ideal can be used to verify the grade conjecture in the case that $\mathrm{grade}_R\, M = 1$.

Proposition 4.23. *If R is Noetherian and local and M is a finitely generated module with $\mathrm{pd}_R\, M < \infty$ and $\mathrm{grade}_R\, M \geq 1$, then*

$$\mathrm{grade}_R\, M = 1 \iff G_R(M) \neq R \iff \dim_R M = \dim R - 1.$$

Proof: The first bi-implication is stated in Theorem 4.20. In addition, since $\mathrm{grade}_R\, M + \dim_R M \leq \dim R$, the equivalences of Theorem 0.8 ensure that $\dim_R M = \dim R - 1$ implies $\mathrm{grade}_R\, M = 1$. For the remaining part of the proposition, we assume that $\mathrm{grade}_R\, M = 1$ and want to show that $\dim_R M = \dim R - 1$. But we cannot have $\dim_R M = \dim R$ according to Theorem 0.8, and since $\mathrm{Supp}_R(R/G_R(M)) \subseteq \mathrm{Supp}_R\, M$ and $G_R(M)$ is a proper principal ideal according to (i), (iii) and (v) of Theorem 4.20, it follows from Theorem 0.12 that

$$\dim R - 1 \leq \dim_R(R/G_R(M)) \leq \dim_R M.$$

Consequently, $\dim_R M = \dim R - 1$. □

Chapter 5

Local Chern characters

The previous chapters have associated with a complex X an element $[X]$ of an Abelian group through a map that is additive on short exact sequences of complexes. This chapter does something similar, associating with each complex X a family $\mathrm{ch}^R(X)$ of linear maps between $\mathbb{Q}$-vector spaces in a way that, in some sense, is additive on short exact sequences. The family $\mathrm{ch}^R(X)$ is the *local Chern character* of X.

The very definition of local Chern characters is by far too intricate to be included in this thesis, whereas the many nice properties of local Chern characters can easily be understood. Consequently, almost none of the theorems here are presented with proof; for more details, the reader is referred to [Rob98].

5.1 Chow groups

Throughout this section, R is assumed to be Noetherian.

The spectrum of a ring can be endowed with a topology known as the *Zariski topology*. The closed subsets of $\operatorname{Spec} R$ are the sets $V_R(I)$ for all ideals I in R. This does, indeed, define a topology on $\operatorname{Spec} R$, since $\operatorname{Spec} R = V_R(0)$, $\emptyset = V_R(R)$,

$$V_R(I_1) \cup \cdots \cup V_R(I_n) = V_R(I_1 \cdots I_n)$$

whenever $I_1, \ldots, I_n$ are ideals of R, and

$$\bigcap_j V_R(I_j) = V_R(\sum_j I_j)$$

whenever $(I_j)_j$ is a family of ideals of R. (The sum of ideals in the last equation is the set of all *finite* sums of elements from each of the ideals.)

Given an element $\mathfrak{p}$ of $\operatorname{Spec} R$, the closure of $\{\mathfrak{p}\}$ in the Zariski topology is the set $V_R(\mathfrak{p})$; this is indeed a closed set containing $\mathfrak{p}$, and it must be the smallest such set, for if $V_R(I)$ is any other closed set containing $\mathfrak{p}$, then we must have $I \subseteq \mathfrak{p}$ and thereby $V_R(\mathfrak{p}) \subseteq V_R(I)$. It follows that the only closed points of $\operatorname{Spec} R$ are the maximal ideals.

If $\mathfrak{V}$ is a closed subset of $\operatorname{Spec} R$, the *dimension* of $\mathfrak{V}$ is the number $\dim_R \mathfrak{V}$ defined by $\dim_R \mathfrak{V} = \dim_R R/I$ for an ideal I such that $V_R(I) = \mathfrak{V}$: that is, $\dim_R \mathfrak{V}$ is the supremum of the lengths of chains of prime ideals in $\mathfrak{V}$.

The support of a finitely generated module M is, as mentioned in the preliminaries, the closed subset $\operatorname{Supp}_R M = V_R(\operatorname{Ann}_R M)$ of $\operatorname{Spec} R$. We extend the definition of support to complexes, defining the *support* of a complex $X \in \mathcal{C}^R_\square(\mathrm{f})$ to be the subset

$$\operatorname{Supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \,|\, X_\mathfrak{p} \text{ is exact}\}.$$

If $X \in \mathcal{C}^R_\square(\mathrm{f,P})$ and $Y \in \mathcal{C}^R_\square(\mathrm{f})$, then $\operatorname{Supp}_R(X \otimes_R Y) = \operatorname{Supp}_R X \cap \operatorname{Supp}_R Y$ (cf. [Fox98, Lemma 14.5]). If $0 \to X \to Y \to Z \to 0$ is a short exact sequence of complexes in $\mathcal{C}^R_\square(\mathrm{f})$, then $\operatorname{Supp}_R Y \subseteq \operatorname{Supp}_R X \cup \operatorname{Supp}_R Z$; this is apparent from the long exact sequence of homology modules.

We define the *annihilator* of $X \in \mathcal{C}^R_\square(\mathrm{f})$ to be the product of the ideals $\operatorname{Ann}_R \mathrm{H}_\ell(X)$, among which all but finitely many are equal to R: that is,

$$\operatorname{Ann}_R X = \cdots (\operatorname{Ann}_R \mathrm{H}_{\ell+1}(X))(\operatorname{Ann}_R \mathrm{H}_\ell(X)) \cdots .$$

It follows that $\operatorname{Supp}_R X = V_R(\operatorname{Ann}_R X)$; in particular, $\operatorname{Supp}_R X$ is closed in the Zariski topology.

Definition 5.1. If $\mathfrak{V}$ is a closed subset of $\operatorname{Spec} R$ and i is an integer, let $\mathfrak{V}_i = \{\mathfrak{p} \in \mathfrak{V} \,|\, \dim_R R/\mathfrak{p} = i\}$ and let $Z^R_i(\mathfrak{V})$ denote the free Abelian group whose basis consists of the symbols $[R/\mathfrak{p}]$ for $\mathfrak{p} \in \mathfrak{V}_i$. In particular, $Z^R_i(\emptyset)$ is the trivial group.

The notation "$[R/\mathfrak{p}]$" is purely symbolic: even if $R/\mathfrak{p} \cong R/\mathfrak{q}$ for prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, this does not mean that $[R/\mathfrak{p}]$ is identical with $[R/\mathfrak{q}]$. In case R is an integral domain, we shall, however, deviate slightly from this policy of strict symbolism, writing $[R]$ instead of $[R/0]$.

Let M be a finitely generated module with $\dim_R M \leq i$ and let $\mathfrak{p} \in \mathfrak{V}_i$ for some closed subset $\mathfrak{V}$ of $\operatorname{Spec} R$. Recall that the localization $M_\mathfrak{p}$ is nontrivial if and only if $\mathfrak{p}$ contains $\operatorname{Ann}_R M$, in which case $\mathfrak{p}$ is minimal in $V_R(\operatorname{Ann}_R M)$ and $M_\mathfrak{p}$ has finite length as an $R_\mathfrak{p}$-module according to Theorem 0.10. As mentioned in the preliminaries, there are only finitely many minimal primes over an ideal when R is Noetherian, so it follows that the sum $\sum_{\mathfrak{p} \in \mathfrak{V}_i} \operatorname{length}_{R_\mathfrak{p}}(M_\mathfrak{p})[R/\mathfrak{p}]$ is finite and therefore determines a well-defined element of $Z^R_i(\mathfrak{V})$.

Definition 5.2. Let $\mathfrak{V}$ be a given closed subset of $\operatorname{Spec} R$. For a finitely generated module M with $\dim_R M = i$, we define an element $[M]^\mathfrak{V}$ in $Z^R_i(\mathfrak{V})$ by

$$[M]^\mathfrak{V} = \sum_{\mathfrak{p} \in \mathfrak{V}_i} \operatorname{length}_{R_\mathfrak{p}}(M_\mathfrak{p})[R/\mathfrak{p}].$$

Furthermore, for a prime $\mathfrak{q} \in \mathfrak{V}_{i+1}$ and an element $x \in R$, we define an element $\operatorname{div}^\mathfrak{V}(\mathfrak{q}, x)$ by

$$\operatorname{div}^\mathfrak{V}(\mathfrak{q}, x) = \begin{cases} [R/(\mathfrak{q} + \langle x \rangle)]^\mathfrak{V}, & \text{if } x \notin \mathfrak{q}, \\ 0, & \text{if } x \in \mathfrak{q}. \end{cases}$$

The quotient of the group $Z_i^R(\mathfrak{V})$ by the subgroup generated by the elements $\operatorname{div}^{\mathfrak{V}}(\mathfrak{q}, x)$ for all $\mathfrak{q} \in \mathfrak{V}_{i+1}$ and $x \in R$ is denoted by $A_i^R(\mathfrak{V})$. Two elements of $Z_i^R(\mathfrak{V})$ are said to be *rationally equivalent* if they represent the same element in $A_i^R(\mathfrak{V})$. The *Chow group* $A^R(\mathfrak{V})$ of $\mathfrak{V}$ is the direct sum of the $A_i^R(\mathfrak{V})$'s. The *rational Chow group* is the group (or $\mathbb{Q}$-vector space) $A^R(\mathfrak{V})_{\mathbb{Q}} = A^R(\mathfrak{V}) \otimes_{\mathbb{Z}} \mathbb{Q}$ which is the direct sum of the groups $A_i^R(\mathfrak{V})_{\mathbb{Q}} = A_i^R(\mathfrak{V}) \otimes_{\mathbb{Z}} \mathbb{Q}$. The image of an element $[R/\mathfrak{p}] \in Z_i^R(\mathfrak{V})$ in $A^R(\mathfrak{V})_{\mathbb{Q}}$ shall also be denoted by $[R/\mathfrak{p}]$. Likewise, the image of $[M]^{\mathfrak{V}} \in Z_i^R(\mathfrak{V})$ in $A^R(\mathfrak{V})_{\mathbb{Q}}$ shall also be denoted by $[M]^{\mathfrak{V}}$, and the i'th component of such an element (which is nontrivial only if $i = \dim_R M$) shall be denoted $[M]_i^{\mathfrak{V}}$.

We shall only consider *rational* Chow groups, so from now on, these are simply referred to as *Chow groups*. The i'th component $A_i^R(\mathfrak{V})_{\mathbb{Q}}$ of a Chow group $A^R(\mathfrak{V})_{\mathbb{Q}}$ is nontrivial only if $\mathfrak{V}$ contains prime ideals $\mathfrak{p}$ with $\dim_R R/\mathfrak{p} = i$; hence $A^R(\mathfrak{V})_{\mathbb{Q}}$ is concentrated in degrees $0, \ldots, \dim_R \mathfrak{V}$ (or $0, 1, \ldots$ if $\dim_R \mathfrak{V} = \infty$).

Note that, if we consider the module $R/\mathfrak{p}$ for a prime $\mathfrak{p}$ contained in $\mathfrak{V}_i$ for some closed subset $\mathfrak{V} \subseteq \operatorname{Spec} R$, then $[R/\mathfrak{p}]^{\mathfrak{V}} = [R/\mathfrak{p}]$ in $Z_i^R(\mathfrak{V})$ and hence in $A^R(\mathfrak{V})_{\mathbb{Q}}$. Also note that, by definition, we must have that $A^R(\emptyset)_{\mathbb{Q}}$ is equal to the trivial group, and that $A^R(\{\mathfrak{m}\})_{\mathbb{Q}}$ for any maximal ideal $\mathfrak{m}$ is equal to the group generated as a $\mathbb{Q}$-vector space by $[R/\mathfrak{m}]$: that is, $A^R(\{\mathfrak{m}\})_{\mathbb{Q}} \cong \mathbb{Q}$.

If $\mathfrak{V} \subseteq \mathfrak{W}$ is an inclusion of closed subsets of $\operatorname{Spec} R$, there is a natural homomorphism $\iota_i^{\mathfrak{V},\mathfrak{W}} \colon A_i^R(\mathfrak{V})_{\mathbb{Q}} \to A_i^R(\mathfrak{W})_{\mathbb{Q}}$ given by $\iota_i^{\mathfrak{V},\mathfrak{W}}([R/\mathfrak{p}]) = [R/\mathfrak{p}]$ for all $\mathfrak{p} \in \mathfrak{V}_i$. Note here that (similarly to the case of Grothendieck groups) the two $[R/\mathfrak{p}]$'s are different: one is an element of $A_i^R(\mathfrak{V})_{\mathbb{Q}}$, whereas the other is an element of $A_i^R(\mathfrak{W})_{\mathbb{Q}}$. The fact that $\iota_i^{\mathfrak{V},\mathfrak{W}}$ is induced by the inclusion map $Z_i^R(\mathfrak{V}) \to Z_i^R(\mathfrak{W})$ does not mean that $\iota_i^{\mathfrak{V},\mathfrak{W}}$ is injective—it only ensures that it is well defined. When the domain $A_i^R(\mathfrak{V})_{\mathbb{Q}}$ and co-domain $A_i^R(\mathfrak{W})_{\mathbb{Q}}$ of a natural map $\iota_i^{\mathfrak{V},\mathfrak{W}}$ are clear from the context, we shall simply denote the natural map by ι; indeed, ι can be thought of as a family of maps $\iota_i^{\mathfrak{V},\mathfrak{W}}$ indexed by nonnegative integers $i \in \mathbb{N}_0$ and closed subsets $\mathfrak{V} \subseteq \mathfrak{W} \subseteq \operatorname{Spec} R$.

5.2 Local Chern characters

Throughout this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$ and quotient field $k = R/\mathfrak{m}$. Furthermore, it is assumed that R is the homomorphic image of a regular local ring Q: that is, $R = Q/I$ for an ideal I of Q.

Suppose $X \in \mathcal{C}_{\square}^R(\text{f,P})$ and let $\mathfrak{X}$ denote the support of X. Recall that $\mathfrak{X}$ is closed. The *local Chern character* of X is a family $\operatorname{ch}^R(X)$ of homomorphisms

$$\operatorname{ch}_i^R(X)^{j,\mathfrak{V}} \colon A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{V} \cap \mathfrak{X})_{\mathbb{Q}}$$

for all $i, j \in \mathbb{N}_0$ and all closed subsets $\mathfrak{V}$ of $\operatorname{Spec} R$. Note that, since the group $A_j^R(\mathfrak{V})_{\mathbb{Q}}$ is nontrivial only if $0 \leq j \leq \dim_R \mathfrak{V}$, the map $\operatorname{ch}_i^R(X)^{j,\mathfrak{V}}$ is nontrivial only if $i \leq j \leq \dim_R \mathfrak{V}$ and $j - i \leq \dim_R \mathfrak{V} \cap \mathfrak{X}$. For convenience, we define $\operatorname{ch}_i^R(X)^{j,\mathfrak{V}}$ to be the zero map $A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{V} \cap \mathfrak{X})_{\mathbb{Q}}$ when i is a negative integer.

For all $i \in \mathbb{N}_0$ and all closed subsets $\mathfrak{V}$ of $\operatorname{Spec} R$, we define $\operatorname{ch}_i^R(X)^{\mathfrak{V}}$ to be the homomorphism $A^R(\mathfrak{V})_{\mathbb{Q}} \to A^R(\mathfrak{V} \cap \mathfrak{X})_{\mathbb{Q}}$ given in degree j by $\operatorname{ch}_i^R(X)^{j,\mathfrak{V}}$. In this way, $\operatorname{ch}_i^R(X)^{\mathfrak{V}}$ is a homomorphism $A^R(\mathfrak{V})_{\mathbb{Q}} \to A^R(\mathfrak{V}\cap\mathfrak{X})_{\mathbb{Q}}$ of degree $-i$. The *i'th local Chern character* of X is the family $\operatorname{ch}_i^R(X)$ of homomorphisms $\operatorname{ch}_i^R(X)^{\mathfrak{V}}$ for all closed subsets $\mathfrak{V}$ of $\operatorname{Spec} R$. In general, we can think of the i'th local Chern character as an operator of degree $-i$ on Chow groups. For any closed subset $\mathfrak{V}$ of $\operatorname{Spec} R$, we define $\operatorname{ch}^R(X)^{\mathfrak{V}}$ to be the homomorphism $A^R(\mathfrak{V})_{\mathbb{Q}} \to A^R(\mathfrak{V} \cap \mathfrak{X})_{\mathbb{Q}}$ whose $(j, j-i)$-entry is $\operatorname{ch}_i^R(X)^{j,\mathfrak{V}}$.

When given a local Chern character $\operatorname{ch}^R(X)$ it determines a homomorphism $\operatorname{ch}_i^R(X)^{j,\mathfrak{V}}\colon A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{V}\cap\mathfrak{X})_{\mathbb{Q}}$ for any choice of $i, j \in \mathbb{N}_0$ and closed subset $\mathfrak{V}$ of $\operatorname{Spec} R$. We shall often compose $\operatorname{ch}_i^R(X)^{j,\mathfrak{V}}$ with a natural homomorphism $\iota_{j-i}^{\mathfrak{V}\cap\mathfrak{X},\mathfrak{W}}$, thereby obtaining a map $A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{W})_{\mathbb{Q}}$ for any choice of $i, j \in \mathbb{N}_0$ and closed subsets $\mathfrak{V}, \mathfrak{W}$ of $\operatorname{Spec} R$ with $\mathfrak{V} \cap \mathfrak{X} \subseteq \mathfrak{W}$. Whenever it is clear from the context what the domain and co-domain are (or if the choice of these is unimportant), we shall simply denote such a map by $\operatorname{ch}^R(X)$.

We can add local Chern characters in the following way. Suppose that X and Y are complexes in $\mathcal{C}_{\square}^R$(f,P) with supports $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. The sum $\operatorname{ch}^R(X) + \operatorname{ch}^R(Y)$ is then defined to be a family of maps

$$(\operatorname{ch}^R(X) + \operatorname{ch}^R(Y))_i^{j,\mathfrak{V}}\colon A_i^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{V} \cap (\mathfrak{X} \cup \mathfrak{Y}))_{\mathbb{Q}}$$

for all $i, j \in \mathbb{N}_0$ and closed subsets $\mathfrak{V}$ of $\operatorname{Spec} R$, given by

$$(\operatorname{ch}^R(X) + \operatorname{ch}^R(Y))_i^{j,\mathfrak{V}} = \operatorname{ch}_i^R(X) + \operatorname{ch}_i^R(Y),$$

in which we consider $\operatorname{ch}_i^R(X)$ and $\operatorname{ch}_i^R(Y)$ as maps $A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{V}\cap(\mathfrak{X}\cup\mathfrak{Y}))_{\mathbb{Q}}$. It is clear that the addition is commutative. At first, calling this construction an addition of local Chern characters may seem odd, since it is not clear that the sum of two local Chern characters is itself a local Chern character; this, however, is the case, as follows from Proposition 5.3(iii) below, which shows that $\operatorname{ch}^R(X) + \operatorname{ch}^R(Y) = \operatorname{ch}^R(X \oplus Y)$.

The local Chern character $\operatorname{ch}^R(0)$ of the zero complex is a neutral element with respect to the addition described above. We shall denote this local Chern character by 0 and refer to it as the *trivial local Chern character*.

A kind of inverse with respect to the addition also exists. Suppose that $X \in \mathcal{C}_{\square}^R$(f,P) has support $\mathfrak{X}$. Then $-\operatorname{ch}^R(X)$ is defined to be a family of maps

$$(-\operatorname{ch}^R(X))_i^{j,\mathfrak{V}}\colon A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{V} \cap \mathfrak{X})_{\mathbb{Q}}$$

for all closed subsets $\mathfrak{V} \subseteq \operatorname{Spec} R$ and $i, j \in \mathbb{N}_0$, given by

$$(-\operatorname{ch}^R(X))_i^{j,\mathfrak{V}} = -\operatorname{ch}_i^R(X)^{j,\mathfrak{V}}.$$

Again, it is not clear that $-\operatorname{ch}^R(X)$ is itself a local Chern character; this, however, is in fact the case, as follows from Proposition 5.3(iii) and (iv) below, which show that $-\operatorname{ch}^R(X) = \operatorname{ch}^R(\Sigma X) - \operatorname{ch}^R(\mathcal{M}(\mathbb{1}_X)) = \operatorname{ch}^R(\Sigma X)$. It is tempting to think that $-\operatorname{ch}^R(X)$ is an inverse to $\operatorname{ch}^R(X)$ under the addition so that their sum is equal to the trivial local Chern character, but this is not entirely true. Although the sum $\operatorname{ch}^R(X) + (-\operatorname{ch}^R(X))$ is a family of zero maps $A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{V} \cap \mathfrak{X})_{\mathbb{Q}}$, this family is not the same as the family of zero maps $A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\emptyset)_{\mathbb{Q}}$ defined by the trivial local Chern character. By abuse of notation, however, we shall allow ourselves to write $\operatorname{ch}_i^R(X) = 0$ whenever $\operatorname{ch}_i^R(X)$ is a family of zero maps, regardless of whether the support of X is empty.

We can multiply local Chern characters in the following way. Suppose that X and Y are complexes in $\mathcal{C}_{\square}^R(\mathrm{f,P})$ with supports $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. The product $\operatorname{ch}^R(X)\operatorname{ch}^R(Y)$ is then defined to be a family of maps

$$(\operatorname{ch}^R(X)\operatorname{ch}^R(Y))_i^{j,\mathfrak{V}} \colon A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-i}^R(\mathfrak{V} \cap \mathfrak{X} \cap \mathfrak{Y})_{\mathbb{Q}}$$

for all $i, j \in \mathbb{N}_0$ and closed subsets $\mathfrak{V}$ of $\operatorname{Spec} R$, given by

$$((\operatorname{ch}^R(X)\operatorname{ch}^R(Y))_i^{j,\mathfrak{V}} = \sum_{m+n=i} \operatorname{ch}_m^R(X)\operatorname{ch}_n^R(Y),$$

in which we consider $\operatorname{ch}_n^R(Y)$ as a map $A_j^R(\mathfrak{V})_{\mathbb{Q}} \to A_{j-n}^R(\mathfrak{V} \cap \mathfrak{Y})_{\mathbb{Q}}$ and $\operatorname{ch}_m^R(X)$ as a map $A_{j-n}^R(\mathfrak{V} \cap \mathfrak{Y})_{\mathbb{Q}} \to A_{j-m-n}^R(\mathfrak{V} \cap \mathfrak{X} \cap \mathfrak{Y})_{\mathbb{Q}}$. Once again, calling this construction a multiplication may seem odd, since it is not clear that the product of two local Chern characters is itself a local Chern character; this, however, is indeed the case, as follows from Proposition 5.3(iv) below, which states that $\operatorname{ch}^R(X)\operatorname{ch}^R(Y) = \operatorname{ch}^R(X \otimes_R Y)$. This said, it is also clear that the local Chern character $\operatorname{ch}^R(R)$ of the complex R concentrated in degree 0 is a neutral element for the multiplication of local Chern characters.

We now list some of the properties of local Chern characters (cf. [Rob85] and [Rob98, Theorem 12.1.2 and Corollary 11.4.1]).

Proposition 5.3. *If X, Y and Z are complexes in $\mathcal{C}_{\square}^R(\mathrm{f,P})$ with supports $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$, respectively, then the following hold.*

(i) $\operatorname{ch}^R(X)\iota = \iota \operatorname{ch}^R(X)$*: that is, if* $\mathfrak{U} \subseteq \mathfrak{V} \subseteq \operatorname{Spec} R$ *are closed and* $i, j \in \mathbb{N}_0$*, then the diagram*

$$\begin{array}{ccc} A_j^R(\mathfrak{U})_{\mathbb{Q}} & \xrightarrow{\operatorname{ch}^R(X)} & A_{j-i}^R(\mathfrak{U} \cap \operatorname{Supp}_R X)_{\mathbb{Q}} \\ \downarrow \iota & & \downarrow \iota \\ A_j^R(\mathfrak{V})_{\mathbb{Q}} & \xrightarrow{\operatorname{ch}^R(X)} & A_{j-i}^R(\mathfrak{V} \cap \operatorname{Supp}_R X)_{\mathbb{Q}} \end{array}$$

is commutative.

(ii) $\operatorname{ch}^R_m(X)\operatorname{ch}^R_n(Y) = \operatorname{ch}^R_n(Y)\operatorname{ch}^R_m(X)$*: that is, if* $\mathfrak{V} \subseteq \operatorname{Spec} R$ *is closed and* $j, m, n \in \mathbb{N}_0$*, then the diagram*

$$\begin{CD}
A^R_j(\mathfrak{V})_{\mathbb{Q}} @>{\operatorname{ch}^R(X)}>> A^R_{j-m}(\mathfrak{V}\cap\mathfrak{X})_{\mathbb{Q}} \\
@V{\operatorname{ch}^R(Y)}VV @VV{\operatorname{ch}^R(Y)}V \\
A^R_{j-n}(\mathfrak{V}\cap\mathfrak{Y})_{\mathbb{Q}} @>{\operatorname{ch}^R(X)}>> A^R_{j-m-n}(\mathfrak{V}\cap\mathfrak{X}\cap\mathfrak{Y})_{\mathbb{Q}}
\end{CD}$$

is commutative. In particular, $\operatorname{ch}^R(X)\operatorname{ch}^R(Y) = \operatorname{ch}^R(Y)\operatorname{ch}^R(X)$.

(iii) $\operatorname{ch}^R(Y) = \operatorname{ch}^R(X) + \operatorname{ch}^R(Z)$ *whenever there is a short exact sequence* $0 \to X \to Y \to Z \to 0$*: that is, if* $\mathfrak{V} \subseteq \operatorname{Spec} R$ *is closed and* $i, j \in \mathbb{N}_0$*, then in the diagram*

$$\begin{array}{ccccc}
 & & A^R_{j-i}(\mathfrak{V}\cap\mathfrak{X})_{\mathbb{Q}} & & \\
 & \nearrow^{\operatorname{ch}^R(X)} & & \searrow^{\iota} & \\
A^R_j(\mathfrak{V})_{\mathbb{Q}} & \xrightarrow{\operatorname{ch}^R(Y)} & A^R_{j-i}(\mathfrak{V}\cap\mathfrak{Y})_{\mathbb{Q}} & \xrightarrow{\iota} & A^R_{j-i}(\mathfrak{V}\cap(\mathfrak{X}\cup\mathfrak{Z}))_{\mathbb{Q}} \\
 & \searrow_{\operatorname{ch}^R(Z)} & & \nearrow_{\iota} & \\
 & & A^R_{j-i}(\mathfrak{V}\cap\mathfrak{Z})_{\mathbb{Q}} & &
\end{array}$$

the sum of the upper and lower maps equals the middle map.

(iv) $\operatorname{ch}^R(X) = 0$ *whenever* X *is exact.*

(v) $\operatorname{ch}^R(X \otimes_R Y) = \operatorname{ch}^R(X)\operatorname{ch}^R(Y)$*: that is, if* $\mathfrak{V} \subseteq \operatorname{Spec} R$ *is closed, then the diagram*

$$\begin{array}{ccc}
A^R(\mathfrak{V})_{\mathbb{Q}} & \xrightarrow{\operatorname{ch}^R(Y)} & A^R(\mathfrak{V}\cap\mathfrak{Y})_{\mathbb{Q}} \\
 & \searrow_{\operatorname{ch}^R(X\otimes_R Y)} & \downarrow{\scriptstyle \operatorname{ch}^R(X)} \\
 & & A^R(\mathfrak{V}\cap\mathfrak{X}\cap\mathfrak{Y})_{\mathbb{Q}}
\end{array}$$

is commutative. In particular, if $i, j \in \mathbb{N}_0$*, then*

$$\operatorname{ch}^R_i(X \otimes_R Y)^{j,\mathfrak{V}} = \sum_{m+n=i} \operatorname{ch}^R_m(X)^{j-n,\mathfrak{V}\cap\mathfrak{Y}} \operatorname{ch}^R_n(Y)^{j,\mathfrak{V}}. \qquad \square$$

Part (i) of Proposition 5.3 states that, when a local Chern character $\operatorname{ch}^R_i(X)^{\mathfrak{V}}$ is applied to an element η that is a linear combination of $[R/\mathfrak{p}]$'s, it does not really matter in what Chow group we choose to place η. To explain this in greater detail, let X be a complex in $\mathcal{C}^R_\square$(f,P) with support $\mathfrak{X}$, and let η be a formal sum in the form

$$\eta = n_1[R/\mathfrak{p}_1] + \cdots + n_t[R/\mathfrak{p}_t]$$

for $n_1, \dots, n_t \in \mathbb{Z}$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_t \in \operatorname{Spec} R$. For any closed subset $\mathfrak{W}$ of $\operatorname{Spec} R$ with $V_R(\mathfrak{p}_1 \cdots \mathfrak{p}_t) \cap \mathfrak{X} \subseteq \mathfrak{W}$, we can then let

$$\operatorname{ch}^R(X)(\eta) \overset{\text{def}}{=} \operatorname{ch}^R(X)^{\mathfrak{V}}(\eta) \in A^R(\mathfrak{W})_{\mathbb{Q}}$$

and

$$\operatorname{ch}_i^R(X)(\eta) \overset{\text{def}}{=} \operatorname{ch}_i^R(X)^{\mathfrak{V}}(\eta) \in A^R(\mathfrak{W})_{\mathbb{Q}},$$

for *any* choice of closed subset $\mathfrak{V}$ of $\operatorname{Spec} R$ containing $\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ and satisfying the condition that $\mathfrak{V} \cap \mathfrak{X} \subseteq \mathfrak{W}$. (On the right side of the equations, we have considered η as an element in $A^R(\mathfrak{V})_{\mathbb{Q}}$.) This is done completely without reference to $\mathfrak{V}$; the only thing that matters is that we specify $\mathfrak{W}$ and that $\mathfrak{W}$ is sufficiently large.

Where the local Chern characters are *operators* on Chow groups associated to every bounded complex of free and finitely generated modules, there is an *element* of a Chow group associated to every bounded complex of finitely generated (not necessarily free) modules. Given a complex X in $\mathcal{C}_{\square}^R(\mathrm{f})$ with support $\mathfrak{X}$, the *Todd class* of X is an element $\tau^R(X)$ in the Chow group $A^R(\mathfrak{X})_{\mathbb{Q}}$. The component in degree i of $\tau^R(X)$ is denoted by $\tau_i^R(X)$. By applying the natural homomorphism ι, we may choose to consider $\tau^R(X)$ as an element in $A^R(\mathfrak{V})_{\mathbb{Q}}$ for any closed subset $\mathfrak{V}$ of $\operatorname{Spec} R$ with $\mathfrak{X} \subseteq \mathfrak{V}$.

In order to define the Todd classes, one needs to know that R is the homomorphic image of a regular local ring; this is why we have made this assumption throughout the present section. Todd classes are defined in terms of local Chern characters, whose definition we have avoided, so there is no reason to present the actual definition of Todd classes here. Instead, we list a few of their many nice properties (cf. [Rob87, page 8] and [Rob98, Corollary 12.4.3 and Proposition 12.4.4]).

Proposition 5.4. *If X, Y and Z are complexes in $\mathcal{C}_{\square}^R(\mathrm{f})$ with supports $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$, respectively, then the following hold.*

(i) $\tau^R(Y) = \tau^R(X) + \tau^R(Z)$ *in* $A^R(\mathfrak{X} \cup \mathfrak{Z})_{\mathbb{Q}}$ *whenever there is a short exact sequence* $0 \to X \to Y \to Z \to 0$.

(ii) $\tau^R(X) = 0$ *whenever X is exact.*

(iii) $\tau^R(X \otimes_R Y) = \operatorname{ch}^R(X)(\tau^R(Y))$ *whenever* $X \in \mathcal{C}_{\square}^R(\mathrm{f,P})$.

(iv) $\tau_i^R(X) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell [\mathrm{H}_\ell(X)]_i^{\mathfrak{X}}$ *whenever the homology modules of X have dimensions not exceeding i.*

(v) $\tau^R(R) = [R]^{\operatorname{Spec} R}$ *whenever R is a complete intersection.* □

If X and Y are complexes in $\mathcal{C}^R_\square(\mathrm{f})$ such that there is a homology isomorphism $\phi\colon X \xrightarrow{\simeq} Y$, then their Todd classes are equal; this follows by applying (i) and (ii) of the proposition to the short exact sequences $0 \to Y \to \mathcal{M}(\phi) \to \Sigma X \to 0$ and $0 \to X \to \mathcal{M}(\mathbb{1}_X) \to \Sigma X \to 0$.

Part (iii) of the proposition is known as the *local Riemann–Roch formula.* From this follows that, whenever $X \in \mathcal{C}^R_\square(\mathrm{f,P})$,

$$\tau^R(X) = \tau^R(X \otimes_R R) = \mathrm{ch}^R(X)(\tau^R(R)). \tag{5.1}$$

Combining this with the observation from the previous paragraph we find that, if $X \in \mathcal{C}^R_\square(\mathrm{f,P})$ is a projective resolution of $M \in \mathcal{C}^R_0(\mathrm{f})$, then

$$\tau^R(M) = \mathrm{ch}^R(X)(\tau^R(R)).$$

According to part (v) of the proposition, this is particularly interesting when R is a complete intersection.

From part (iv) of the proposition, it follows that if $X \in \mathcal{C}^R_\square(\mathrm{f|l})$,

$$\tau^R(X) = \mathcal{H}([X])[R/\mathfrak{m}] \tag{5.2}$$

in $A^R(\{\mathfrak{m}\})_\mathbb{Q}$; here $\mathcal{H}$ denotes the homomorphism $G^R_\square(\mathrm{f|l}) \to G^R_0(\mathrm{l})$ from Theorem 2.11 (which is applicable since $G^R_0(\mathrm{l})$ contains the kernels of all its homomorphisms) in which we have used the identification of $G^R_0(\mathrm{l})$ with $\mathbb{Z}$ as established in Example 2.16; thus $\mathcal{H}([X]) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{length}_R \mathrm{H}_\ell(X)$. From part (iv), it also follows that, if $M \in \mathcal{C}^R_0(\mathrm{f})$, then

$$\tau^R(M) = [M]^{\operatorname{Supp}_R M} + \text{terms of lower degree.} \tag{5.3}$$

This chapter concludes by describing how the zeroth and first local Chern characters are closely related to the Euler characteristic and the MacRae ideal.

Proposition 5.5. *If $X \in \mathcal{C}^R_\square(\mathrm{f,P})$, then $\mathrm{ch}^R_0(X)$ is multiplication by $\chi^R(X)$: that is, if $\mathfrak{V} \subseteq \operatorname{Spec} R$ is closed, $j \in \mathbb{N}_0$ and $\mathfrak{X}$ is the support of X, then the map $\mathrm{ch}^R_0(X)^{j,\mathfrak{V}}\colon A^R_j(\mathfrak{V})_\mathbb{Q} \to A^R_j(\mathfrak{V}\cap\mathfrak{X})_\mathbb{Q}$ is given by*

$$\mathrm{ch}^R_0(X)^{j,\mathfrak{V}}[R/\mathfrak{p}] = \chi^R(X)[R/\mathfrak{p}]^{\mathfrak{V}\cap\mathfrak{X}}$$

for all $\mathfrak{p} \in \mathfrak{V}_j$.

Proof: Note first that according to (5.3), $[R/\mathfrak{p}] = \tau^R_j(R/\mathfrak{p})$ and $\tau^R_{i+j}(R/\mathfrak{p}) = 0$ for all $i > 0$. Thus, since $\operatorname{Supp}_R R/\mathfrak{p} = V_R(\mathfrak{p}) \subseteq \mathfrak{V}$, we find in $A^R_j(\mathfrak{V}\cap\mathfrak{X})_\mathbb{Q}$ that

$$\begin{aligned}\mathrm{ch}^R_0(X)^{j,\mathfrak{V}}[R/\mathfrak{p}] &= \mathrm{ch}^R_0(X)^{j,V_R(\mathfrak{p})}[R/\mathfrak{p}]\\ &= \mathrm{ch}^R_0(X)^{j,V_R(\mathfrak{p})}(\tau^R_j(R/\mathfrak{p}))\\ &= \sum_{i\in\mathbb{N}_0}\mathrm{ch}^R_i(X)^{i+j,V_R(\mathfrak{p})}(\tau^R_{i+j}(R/\mathfrak{p}))\end{aligned}$$

is equal to the j'th component of $\operatorname{ch}^R(X)^{V_R(\mathfrak{p})}(\tau^R(R/\mathfrak{p}))$. By the local Riemann–Roch formula (Proposition 5.4(iii)), this is equal to $\tau_j^R(X \otimes_R R/\mathfrak{p})$ in $A_j^R(\mathfrak{V} \cap \mathfrak{X})_{\mathbb{Q}}$, and by Proposition 5.4(iv) this again is equal to $\sum_{\ell\in\mathbb{Z}}(-1)^\ell[\mathrm{H}_\ell(X \otimes_R R/\mathfrak{p})]_j^{V_R(\mathfrak{p})\cap\mathfrak{X}}$. If $\mathfrak{p} \notin \mathfrak{X}$, this is equal to 0 in $A_j^R(\mathfrak{V} \cap \mathfrak{X})_{\mathbb{Q}}$, and the theorem is proved since $[R/\mathfrak{p}]^{\mathfrak{V}\cap\mathfrak{X}} = 0$ in this case. If $\mathfrak{p} \in \mathfrak{X}$, we continue our calculation:

$$\begin{aligned}
\operatorname{ch}_0^R(X)^{j,\mathfrak{V}}[R/\mathfrak{p}] &= \sum_{\ell\in\mathbb{Z}}(-1)^\ell[\mathrm{H}_\ell(X \otimes_R R/\mathfrak{p})]_j^{V_R(\mathfrak{p})\cap\mathfrak{X}} \\
&= \sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{length}_{R_\mathfrak{p}}(\mathrm{H}_\ell(X \otimes_R R/\mathfrak{p}))_\mathfrak{p}[R/\mathfrak{p}] \\
&= \sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{length}_{R_\mathfrak{p}} \mathrm{H}_\ell((X \otimes_R R/\mathfrak{p})_\mathfrak{p})[R/\mathfrak{p}] \\
&= \sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{length}_{R_\mathfrak{p}} \mathrm{H}_\ell(X_\mathfrak{p} \otimes_{R_\mathfrak{p}} (R/\mathfrak{p})_\mathfrak{p})[R/\mathfrak{p}] \\
&= \chi^{R_\mathfrak{p}}(X_\mathfrak{p})[R/\mathfrak{p}] \\
&= \chi^R(X)[R/\mathfrak{p}].
\end{aligned}$$

For the third equality, we have used the fact that localization is an exact functor on modules and therefore induces a functor on complexes that commutes with the homology functor, as described in the preliminaries. The fourth equality is also explained in the preliminaries. This proves the proposition. □

Note that since X is a projective resolution of itself, $\chi^R(X)$ is just the alternating sum of the ranks of the modules in X. Consequently, if $\mathfrak{p}$ is a prime from outside the support of X, then $\chi^R(X) = \chi^{R_\mathfrak{p}}(X_\mathfrak{p}) = 0$. It follows that $\chi^R(X)$ and hence $\operatorname{ch}_0^R(X)$ is nonzero only if $\operatorname{Supp}_R X = \operatorname{Spec} R$. Thus, in this case, $\operatorname{ch}_0^R(X)$ does indeed act as a multiplication, namely multiplication by $\chi^R(X)$ on each Chow group $A^R(\mathfrak{V})_{\mathbb{Q}}$.

Corollary 5.6. *If* $X \in \mathcal{C}_\square^R(\mathrm{f,P})$, *then* $\operatorname{ch}_0^R(X) \neq 0$ *if and only if* $\chi^R(X) \neq 0$.

Proof: "only if" is clear, and "if" follows since $\operatorname{ch}_0^R(X)[R/\mathfrak{m}] = \chi^R(X)[R/\mathfrak{m}]$ is nonzero in $A^R(\{\mathfrak{m}\})_{\mathbb{Q}} \cong \mathbb{Q}$ whenever $\chi^R(X) \neq 0$. □

To describe the first local Chern character, we need to introduce a new operation on Chow groups. The operation is commonly known as *intersection with a divisor*, where the divisor in this case is a principal ideal. The definition is stated below; one should of course verify that it is well defined, that is, that it preserves rational equivalence, but this will be omitted (see instead [Rob98, Proposition 8.8.1]).

Definition 5.7. For every $x \in R$ and every closed subset $\mathfrak{V} \subseteq \operatorname{Spec} R$, we define a homomorphism $\langle x\rangle \cap -\colon A^R(\mathfrak{V})_{\mathbb{Q}} \to A^R(\mathfrak{V} \cap V_R(x))_{\mathbb{Q}}$, referred to as *intersection* *with* $\langle x\rangle$, by

$$\langle x\rangle \cap [R/\mathfrak{p}] = \begin{cases} [R/(\mathfrak{p}+\langle x\rangle)]^{\mathfrak{V}\cap V_R(x)}, & \text{if } x \notin \mathfrak{p}, \\ 0, & \text{if } x \in \mathfrak{p}, \end{cases}$$

for every $\mathfrak{p} \in \mathfrak{V}$.

If $\dim_R R/\mathfrak{p} = i$, then $\langle x\rangle \cap [R/\mathfrak{p}]$ is an element in $A^R_{i-1}(\mathfrak{V}\cap V_R(x))_\mathbb{Q}$; in other words, $\langle x\rangle \cap -$ has degree -1. Note that the element $\langle x\rangle \cap [R/\mathfrak{p}]$ corresponds to the element $\operatorname{div}^{\mathfrak{V}}(\mathfrak{p}, x)$, except that we are evaluating in $A_R(\mathfrak{V}\cap V_R(x))_\mathbb{Q}$ rather than $A_R(\mathfrak{V})_\mathbb{Q}$.

As with local Chern characters, one often composes the intersection map $\langle x\rangle \cap -\colon A^R(\mathfrak{V})_\mathbb{Q} \to A^R(\mathfrak{V}\cap V_R(x))_\mathbb{Q}$ with a natural homomorphism $\iota^{\mathfrak{V}\cap V_R(x),\mathfrak{W}}$, thereby obtaining a map $A^R(\mathfrak{V})_\mathbb{Q} \to A^R(\mathfrak{W})_\mathbb{Q}$ for any choice of closed subset $\mathfrak{W}$ of $\operatorname{Spec} R$ with $\mathfrak{V}\cap V_R(x) \subseteq \mathfrak{W}$. We shall also denote this map by $\langle x\rangle \cap -$.

We can now describe the relation between local Chern characters and the MacRae ideal—as always without proof (see instead [Rob87]).

Proposition 5.8. *If* $X \in \mathcal{C}^R_\square(\text{f,P})$ *is a projective resolution of a module* M *in* $\mathcal{C}^R_0(\text{f,pd,gr} \geq 1)$*, then* $\operatorname{ch}^R_1(X)$ *is intersection with* $G_R(M)$*: that is, if* $\mathfrak{V} \subseteq \operatorname{Spec} R$ *is closed,* $j \in \mathbb{N}_0$ *and* $\mathfrak{X}$ *is the support of* X *(and hence of* M*), then the map* $\operatorname{ch}^R_1(X)^{j,\mathfrak{V}}\colon A^R_j(\mathfrak{V})_\mathbb{Q} \to A^R_{j-1}(\mathfrak{V}\cap\mathfrak{X})_\mathbb{Q}$ *is given by*

$$\operatorname{ch}^R_1(X)^{j,\mathfrak{V}}[R/\mathfrak{p}] = G_R(M) \cap [R/\mathfrak{p}],$$

for all $\mathfrak{p} \in \mathfrak{V}_j$*.* □

Note that, for the proposition to make sense, we are considering $G_R(M)\cap -$ as a map from $A_R(\mathfrak{V})_\mathbb{Q} \to A_R(\mathfrak{V}\cap\mathfrak{X})_\mathbb{Q}$; this is allowed according to Theorem 4.20(v), since $V_R(G_R(M)) \subseteq \mathfrak{X}$.

Corollary 5.9. *If* $X \in \mathcal{C}^R_\square(\text{f,P})$ *is a projective resolution of a module* M *in* $\mathcal{C}^R_0(\text{f,pd,gr} \geq 1)$*, then* $\operatorname{ch}^R_1(X) \neq 0$ *if and only if* $G_R(M) \neq R$*.*

Proof: "only if" is clear. For the other direction, let $x_1 \in R$ be a non-zerodivisor generating $G_R(M)$. Then, according to Proposition 0.15, x_1 is part of a system of parameters $x = (x_1, x_2, \ldots, x_d)$ for R, where $d = \dim R$. From Theorem 0.12, it now follows that $\dim_R /\langle x_2, \ldots, x_d\rangle = 1$, and hence we can choose a prime $\mathfrak{p} \neq \mathfrak{m}$ that is minimal over $\langle x_2, \ldots, x_d\rangle$. Thus, $\mathfrak{p}$ must satisfy the condition that $x_1 \notin \mathfrak{p}$ and $\dim_R R/\mathfrak{p} = 1$; in particular, $V_R(\mathfrak{p}) \cap V_R(x_1) = \{\mathfrak{m}\}$. Applying the first local Chern character of X to $[R/\mathfrak{p}]$, we now get

$$\begin{aligned} \operatorname{ch}^R_1(X)[R/\mathfrak{p}] &= G_R(M) \cap [R/\mathfrak{p}] \\ &= [R/\mathfrak{p} + G_R(M)]^{\{\mathfrak{m}\}} \\ &= \operatorname{length}_R(R/(\mathfrak{p} + G_R(M))[R/\mathfrak{m}], \end{aligned}$$

which is nonzero in $A^R(\{\mathfrak{m}\})_{\mathbb{Q}} \cong \mathbb{Q}$. Consequently $\operatorname{ch}_1^R(X) \neq 0$. □

We have defined the map $\langle x\rangle \cap -$ for every element x of R. One can generalize this definition to include all elements of $(R \setminus \operatorname{Zd} R)^{-1}R$ by letting $\langle r/s\rangle \cap -$ be defined by $\langle r/s\rangle \cap [R/\mathfrak{p}] = \langle r\rangle \cap [R/\mathfrak{p}] - \langle s\rangle \cap [R/\mathfrak{p}]$. This definition depends on r/s only up to multiplication with a unit, so we can define an intersection map for every element of $((R \setminus \operatorname{Zd} R)^{-1}R)/R^*$ by letting $[r/s]_{R^*} \cap -$ be equal to $\langle r/s\rangle \cap -$. It would be interesting to know whether Proposition 5.8 can be generalized to hold for all complexes in $\mathcal{C}_\square^R(\text{f,P}|S\text{-tor})$ for some multiplicative system S with $S \cap \operatorname{Zd} R = \emptyset$, using the determinant $\det_S$ instead of the MacRae ideal: that is, it would be interesting to know whether $\operatorname{ch}_1^R(X)$ is the same as $\det_S X \cap -$ for all $X \in \mathcal{C}_\square^R(\text{f,P}|S\text{-tor})$.

The *Chern grade* of a complex $X \in \mathcal{C}_\square^R(\text{f,P})$ is the number

$$\operatorname{Ch-grade}_R X = \inf\{i \in \mathbb{N}_0 \,|\, \operatorname{ch}_i^R(X) \neq 0\}.$$

It is natural to ask whether the Chern grade, similar to what has been proposed for ordinary grade, satisfies the condition that

$$\operatorname{Ch-grade}_R X = \dim R - \dim_R X,$$

where $\dim_R X = \dim_R(\operatorname{Supp}_R X)$. Corollaries 5.6 and 5.9 together with Theorems 0.8 and Theorem 4.23 verify that this indeed is true whenever X is a projective resolution of a module and $\operatorname{Ch-grade}_R X \leq 1$. However, Dutta, Hochster and McLaughlin's counterexample for the intersection conjectures in the case where only one of the modules has finite projective dimension turns out to serve as a counterexample for this assertion.

Dutta, Hochster and McLaughlin constructed a ring R, a homomorphic image of a regular local ring, with $\dim R = 3$ and two modules M and N with $\dim_R M = 0$ and $\dim_R N = 2$, such that $\chi^R(M,N) = -1$. Let $X \in \mathcal{C}_\square^R(\text{f,P})$ be a projective resolution of M, so that $\chi^R(M,N) = \mathcal{H}([X \otimes_R N])$, where $\mathcal{H}$ is the homomorphism $G_\square^R(\text{f}|\text{l}) \to G_0^R(\text{l})$ from Theorem 2.11 in which we have identified $G_0^R(\text{l})$ with $\mathbb{Z}$ as in Example 2.16. Note that $\operatorname{ch}_0^R(X) = \operatorname{ch}_1^R(X) = 0$ according to Corollaries 5.6 and 5.9 together with Theorems 0.8 and Theorem 4.23. We then find that, in $A^R(\{\mathfrak{m}\})_{\mathbb{Q}}$,

$$\begin{aligned}
-[R/\mathfrak{m}] &= \chi^R(M,N)[R/\mathfrak{m}] \\
&= \mathcal{H}[X \otimes_R N][R/\mathfrak{m}] \\
&= \tau^R(X \otimes_R N) && \text{(according to (5.2))} \\
&= \operatorname{ch}^R(X)(\tau^R(N)) && \text{(using Proposition 5.4(iii))} \\
&= \operatorname{ch}_2^R(X)(\tau_2^R(N)) && \text{(using Corollaries 5.6 and 5.9),}
\end{aligned}$$

and it follows that $\operatorname{ch}_2^R(X) \neq 0$ and thereby $\operatorname{Ch-grade}_R X = 2 \neq \dim R - \dim_R X$.

Chapter 6

The intersection conjectures

This chapter returns to the discussion from Chapter 1 and applies the results established in chapters 2 through 5 to Serre's intersection conjectures.

6.1 Serre's intersection multiplicity

In this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$ and quotient field $k = R/\mathfrak{m}$. Furthermore, M and N denote finitely generated modules with $\operatorname{pd}_R M < \infty$ and $\dim_R(M \otimes_R N) = 0$.

Let us begin by recalling the definition of the intersection multiplicity and the formulation of the intersection conjectures.

Definition 6.1. The *intersection multiplicity* of M and N is the number

$$\chi^R(M, N) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell \operatorname{length}_R \operatorname{Tor}_\ell^R(M, N).$$

This is well defined: the fact that $\operatorname{pd}_R M < \infty$ ensures that $\operatorname{Tor}_\ell^R(M, N)$ is nonzero for only finitely many ℓ, and the fact that $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \operatorname{Supp}_R(M \otimes_R N) = \{\mathfrak{m}\}$ ensures that $\operatorname{Supp}_R(\operatorname{Tor}_\ell^R(M, N)) \subseteq \{\mathfrak{m}\}$: that is, $\operatorname{Tor}_\ell^R(M, N)$ has finite length for all $\ell \in \mathbb{Z}$.

The intersection conjectures. The following hold.

(0) $\dim_R M + \dim_R N \leq \dim R$.

(1) $\chi^R(M, N) \geq 0$. (nonnegativity)

(2) $\chi^R(M, N) \neq 0$ if and only if $\dim_R M + \dim_R N = \dim R$.

Here conditions (1) and (2) can be replaced by

(1′) $\chi^R(M, N) = 0$ if $\dim_R M + \dim_R N < \dim R$. (vanishing)

(2′) $\chi^R(M, N) > 0$ if $\dim_R M + \dim_R N = \dim R$. (positivity)

As mentioned in the introduction, Serre stated the conjectures in the case where R is a regular local ring, and he proved that condition (0) holds in this case. This chapter presents Foxby's proof in [Fox82b] that the intersection conjectures hold when R is Noetherian and local (but not necessarily regular) and either $\operatorname{grade}_R M \leq 1$ or $\dim_R N \leq 1$ and Roberts' proof in [Rob85] that the vanishing conjecture holds when R is a complete intersection (and hence, in particular, when R is a regular local ring).

First we elaborate a bit on the observations made in the introduction. The first thing we notice is the trivial fact that the intersection multiplicity is commutative, in the sense that $\chi^R(M,N) = \chi^R(N,M)$; this follows from the commutativity of $\operatorname{Tor}^R(-,-)$. Defining the intersection multiplicity requires one of the modules to have finite projective dimension, and because of commutativity, we can always assume that M is such a module.

The second thing we notice is that $\chi^R(M,-)$ is additive on short exact sequences: that is, if $0 \to N' \to N \to N'' \to 0$ is an exact sequence of modules on which $\chi^R(M,-)$ is defined, then $\chi^R(M,N) = \chi^R(M,N') + \chi^R(M,N'')$. This follows by an inductive argument on the long exact Tor-sequence

$$\cdots \to \operatorname{Tor}_\ell^R(M,N') \to \operatorname{Tor}_\ell^R(M,N) \to \operatorname{Tor}_\ell^R(M,N'') \to \operatorname{Tor}_{\ell-1}^R(M,N') \to \cdots,$$

using methods similar to the ones used in the proof of Theorem 2.11 that the homomorphism $\mathcal{H}$ is well defined. Similarly (or by commutativity), $\chi^R(-,N)$ is additive on short exact sequences.

Additivity of the intersection multiplicity means that the maps $\chi^R(M,-)$ and $\chi^R(-,N)$ factor through Grothendieck groups of modules on which they can be defined. This allows the theory obtained in chapters 2, 3 and 4 to be applied.

6.2 Applying algebraic K-theory

In this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$ and quotient field $k = R/\mathfrak{m}$. Furthermore, M and N denote finitely generated modules with $\operatorname{pd}_R M < \infty$ and $\dim_R(M \otimes_R N) = 0$.

The proof of the intersection conjectures in the case $\dim_R N = 0$ is not very hard and can easily be established without the use of theory from the preceding chapters. However, we include a proof here that uses some of this theory to give a flavor of how the use of Grothendieck groups can help compute the intersection multiplicity.

Theorem 6.2. *The intersection conjectures hold if* $\dim_R N = 0$.

Proof: The assumption immediately implies that (0) of the intersection conjecture holds. Since $\operatorname{Supp}_R N = \{\mathfrak{m}\}$, $\chi^R(-,N)$ is defined on all modules of finite projective dimension, and hence it factors through the Grothendieck group $G_0^R(\text{f,pd})$. Thus, according to Corollary 3.39,

$$\chi^R(M,N) = \chi^R([M],N) = \chi^R(M)\chi^R([R],N) = \chi^R(M)\operatorname{length}_R N,$$

since, as seen directly from the definition, $\chi^R(R,N) = \operatorname{length}_R N$. We have assumed that $N \neq 0$ (since $\dim_R(M \otimes_R N) = 0$), so $\operatorname{length}_R N$ is nonzero, and the vanishing and positivity conjectures follow immediately from Theorem 0.8. □

We could also prove the conjectures by considering $\chi^R(M,-)$ instead: since $\chi^R(M,-)$ is defined on all modules of finite length, it factors through the Grothendieck group $G_0^R(\text{l})$. Thus, according to Example 2.16,

$$\chi^R(M,N) = \chi^R(M,[N]) = \chi^R(M,[k])\operatorname{length}_R N = \chi^R(M)\operatorname{length}_R N,$$

where we have used the fact that the intersection multiplicity $\chi^R(M,k)$ by definition is equal to the Euler characteristic $\chi^R(M)$ of M.

Corollary 6.3. *The intersection conjectures hold if* $\operatorname{grade}_R M = 0$.

Proof: According to the assumption we must have $\operatorname{Ann}_R M \subseteq \operatorname{Zd} R$ which by Theorem 0.8 implies $\operatorname{Supp}_R M = \operatorname{Spec} R$, and from the assumption that $\dim_R(M \otimes_R N) = 0$ follows $\operatorname{Supp}_R N = \{\mathfrak{m}\}$. Thus, the case that $\operatorname{grade}_R M = 0$ is covered by the case that $\dim_R N = 0$. □

Theorem 6.4 (Foxby). *The intersection conjectures hold if* $\dim_R N = 1$.

Proof: We cannot have $\dim_R M = \dim R$, because this implies $\operatorname{Supp}_R M = \operatorname{Spec} R$ according to Theorem 0.8, contradicting the fact that $\dim_R(M \otimes_R N) = 0$. Thus, we must have $\dim_R M < \dim R$, so (0) of the intersection conjectures holds.

Now, as described in the preliminaries, N has a filtration

$$N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_t = 0$$

in which the factors N_{i-1}/N_i are isomorphic to $R/\mathfrak{q}_i$ for prime ideals $\mathfrak{q}_i$; the set $\{\mathfrak{q}_1, \dots, \mathfrak{q}_t\}$ of prime ideals thus obtained is contained in $\operatorname{Supp}_R N$ and contains $\operatorname{Ass}_R N$. Using additivity of $\chi^R(M,-)$ on each of the exact sequences $0 \to N_i \to N_{i-1} \to R/\mathfrak{q}_i \to 0$, we find that

$$\chi^R(M,N_{i-1}) = \chi^R(M,N_i) + \chi^R(M,R/\mathfrak{q}_i),$$

and it follows inductively that we can write $\chi^R(M,N) = \sum_{i=1}^t \chi^R(M,R/\mathfrak{q}_i)$. For each of the $\mathfrak{q}_i$'s, we have $\dim_R R/\mathfrak{q}_i \leq \dim_R N = 1$, and using Theorem 6.2 we can ignore all terms for which $\dim_R R/\mathfrak{q}_i = 0$. This shows that we can write $\chi^R(M,N)$ as a sum of $\chi^R(M,R/\mathfrak{q}_i)$'s for those $\mathfrak{q}_i$'s with $\dim_R R/\mathfrak{q}_i = 1$. This sum is nonempty since it includes all terms for which $\mathfrak{q}_i$ is minimal over $\operatorname{Ann}_R N$.

To show the vanishing and positivity conjectures, it therefore suffices to consider the case where $N = R/\mathfrak{q}$ for some prime ideal $\mathfrak{q}$.

We can apply $\chi^R(-, R/\mathfrak{q})$ to all finitely generated modules of finite projective dimension whose annihilator contains something from outside $\mathfrak{q}$. In other words, letting $S = R\backslash\mathfrak{q}$, $\chi^R(-, R/\mathfrak{q})$ can be applied to all modules in $\mathcal{C}_0^R(\text{f,pd,}S\text{-tor})$, and hence it factors through $G_0^R(\text{f,pd,}S\text{-tor})$. Thus, since $S^{-1}R = R_\mathfrak{q}$ is local and thereby generalized Euclidean according to Proposition 4.5, it follows from Corollary 4.22 that

$$\begin{aligned}\chi^R(M, R/\mathfrak{q}) &= \chi^R([M], R/\mathfrak{q})\\ &= \chi^R([R/G_R(M)], R/\mathfrak{q})\\ &= \chi^R(R/G_R(M), R/\mathfrak{q}).\end{aligned}$$

This is easily calculated: letting $u \in R$ be a generator for $G_R(M)$ and tensoring the obvious projective resolution for $R/G_R(M) = R/\langle u\rangle$ with $R/\mathfrak{q}$, we obtain the complex

$$0 \longrightarrow R/\mathfrak{q} \stackrel{u}{\longrightarrow} R/\mathfrak{q} \longrightarrow 0.$$

Now, $u/1$ is a representative of $\det_S M$ and is therefore a unit in $S^{-1}R = R_\mathfrak{q}$, and hence u does not belong to $\mathfrak{q}$. Thus, the homology of the above complex is concentrated in degree 0 where it is equal to $R/(G_R(M) + \mathfrak{p})$, and it follows that

$$\chi^R(R/G_R(M), R/\mathfrak{q}) = \operatorname{length}_R R/(G_R(M) + \mathfrak{q}).$$

This is nonzero if and only if $G_R(M) \neq R$: that is, according to Proposition 4.23, if and only if $\dim_R M = \dim R - 1$. This proves the vanishing and positivity conjectures. □

Corollary 6.5. *The intersection conjectures hold if* $\operatorname{grade}_R M = 1$.

PROOF: In this case, according to part (i), (iii) and (v) of Theorem 4.20, $G_R(M)$ is a proper principal ideal such that $\operatorname{Supp}_R R/G_R(M) \subseteq \operatorname{Supp}_R M$, so Theorem 0.12 yields that

$$\begin{aligned}\dim_R N - 1 &\leq \dim_R N/G_R(M)N\\ &= \dim_R(R/G_R(M) \otimes_R N)\\ &\leq \dim_R(M \otimes N),\end{aligned}$$

and hence $\dim_R N \in \{0, 1\}$. These cases are covered by theorems 6.2 and 6.4. □

6.3 Applying local Chern characters

In this section, R is assumed to be a Noetherian, local ring with maximal ideal $\mathfrak{m}$ and quotient field $k = R/\mathfrak{m}$. Furthermore, it is assumed that R is the homomorphic image of a regular local ring Q: that is, $R = Q/I$ for an ideal I of Q. Finally, M and N denote finitely generated modules with $\operatorname{pd}_R M < \infty$ and $\dim_R(M \otimes_R N) = 0$.

We now prove the vanishing conjecture in the case where R is a complete intersection. Although the proof comes out very short and simple below, one should recall the vast amount of work lying behind the propositions (and even definitions) of Chapter 5 whose proofs were elegantly avoided.

Theorem 6.6 (Roberts). *The vanishing conjecture holds whenever R is a complete intersection and N has finite projective dimension.*

PROOF: Suppose that $X, Y \in \mathcal{C}^R_\square(\text{f,P})$ are projective resolutions of M and N, respectively, so that $\chi^R(M, N) = \mathcal{H}([X \otimes_R N])$, where $\mathcal{H}$ is the homomorphism $G^R_\square(\text{f|l}) \to G^R_0(\text{l})$ from Theorem 2.11 in which we have identified $G^R_0(\text{l})$ with $\mathbb{Z}$ as in Example 2.16. We then find that, in $A^R(\{\mathfrak{m}\})_\mathbb{Q}$,

$$\begin{aligned}
\chi^R(M,N)[R/\mathfrak{m}] &= \mathcal{H}[X \otimes_R N][R/\mathfrak{m}] \\
&= \tau^R(X \otimes_R N) && \text{(according to (5.2))} \\
&= \operatorname{ch}^R(X)(\tau^R(N)) && \text{(using Proposition 5.4(iii))} \\
&= \operatorname{ch}^R(X)\operatorname{ch}^R(Y)(\tau^R(R)) && \text{(according to (5.1))} \\
&= \operatorname{ch}^R(X)\operatorname{ch}^R(Y)[R]^{\operatorname{Spec} R} && \text{(using Proposition 5.4(v))} \\
&= \sum_{i+j=d} \operatorname{ch}^R_i(X)\operatorname{ch}^R_j(Y)[R]^{\operatorname{Spec} R},
\end{aligned}$$

where d is the dimension of R. Now, as we saw in Chapter 5, $\operatorname{ch}^R_j(Y)[R]^{\operatorname{Spec} R}$ is trivial for $d - j > \dim_R N$: that is, since $i + j = d$, it must be trivial for $i > \dim_R N$. Similarly, $\operatorname{ch}^R_i(X)[R]^{\operatorname{Spec} R}$ is trivial for $d - i > \dim_R M$: that is, it is trivial for $j > \dim_R M$. It now follows from the commutativity of local Chern characters as stated in Proposition 5.3(ii) that the only nontrivial terms in the sum above are the terms corresponding to $i \leq \dim_R N$ and $j \leq \dim_R M$. But since we are trying to proving the vanishing conjecture, we are assuming that $\dim_R M + \dim_R N < d$, and hence there are no such terms. This proves the vanishing conjecture. □

Since regular implies complete intersection, Theorem 6.6 confirms the vanishing conjecture as originally formulated by Serre. The assumption that R is a complete intersection is only used to deduce that $\tau^R(R) = [R]^{\operatorname{Spec} R}$. A ring satisfying this property is called a *Roberts ring*. Thus, the vanishing conjecture holds whenever R is a Roberts ring and both M and N have finite projective dimensions.

Note that the calculation of $\chi^R(M,N)$ in the proof of Theorem 6.6 holds even without the assumption that $\dim_R M + \dim_R N < d$. Letting $i = \dim_R N$ and $j = \dim_R M$, and assuming that condition (0) of the intersection conjecture holds (which it does for regular local rings, as proven by Serre), we generally find that

$$\chi^R(M,N)[R/\mathfrak{m}] = \mathrm{ch}_i^R(X)\,\mathrm{ch}_j^R(Y)\,[R]^{\operatorname{Spec} R}.$$

Proving the positivity conjecture in the case that R is a complete intersection and N has finite projective dimension is then a matter of showing that the above element is "positive".

We can also use the theory of local Chern characters to prove the intersection conjectures in the cases that $\dim_R N = 0, 1$. The proofs, however, do not appear to be quite as strong as the ones from the preceding section, since we are assuming that R is the homomorphic image of a regular local ring. (But, in fact, the proofs are equally strong, since we are allowed to replace the ring by its completion—a technique that will not be discussed here.)

Let $X \in \mathcal{C}_\square^R(\text{f,P})$ be a projective resolution of M. In the case that $\dim_R N = 0$, the essential calculation in the proof of Theorem 6.2 follows from the following calculation in $A^R(\{\mathfrak{m}\})_\mathbb{Q} \cong \mathbb{Q}$.

$$\begin{aligned}
\chi^R(M,N)[R/\mathfrak{m}] &= \mathcal{H}[X \otimes_R N][R/\mathfrak{m}] \\
&= \tau^R(X \otimes_R N) && \text{(according to (5.2))} \\
&= \mathrm{ch}^R(X)(\tau^R(N)) && \text{(using Proposition 5.4(iii))} \\
&= \mathrm{ch}_0^R(X)[N]^{\{\mathfrak{m}\}} && \text{(according to (5.3))} \\
&= \chi^R(M)[N]^{\{\mathfrak{m}\}} && \text{(using Proposition 5.5)} \\
&= \chi^R(M)\,\mathrm{length}_R N[R/\mathfrak{m}].
\end{aligned}$$

In the case that $\dim_R N = 1$ (by which it follows that $\dim_R M < \dim R$, so that the MacRae ideal of M is defined), we assume, as in the proof of Theorem 6.4, that $N = R/\mathfrak{q}$ for a prime $\mathfrak{q}$, and the essential calculation in the proof of Theorem 6.4 then follows from the following calculation in $A^R(\{\mathfrak{m}\})_\mathbb{Q} \cong \mathbb{Q}$.

$$\begin{aligned}
\chi^R(M,R/\mathfrak{q})[R/\mathfrak{m}] &= \mathcal{H}[X \otimes_R R/\mathfrak{q}][R/\mathfrak{m}] \\
&= \tau^R(X \otimes_R R/\mathfrak{q}) && \text{(according to (5.2))} \\
&= \mathrm{ch}^R(X)(\tau^R(R/\mathfrak{q})) && \text{(using Proposition 5.4(iii))} \\
&= \mathrm{ch}_1^R(X)(\tau_1^R(R/\mathfrak{q})) && \text{(using Corollary 5.6)} \\
&= \mathrm{ch}_1^R(X)[R/\mathfrak{q}] && \text{(according to (5.3))} \\
&= G_R(M) \cap [R/\mathfrak{q}] && \text{(using Proposition 5.8)} \\
&= [R/(\mathfrak{q} + G_R(M))]^{\{\mathfrak{m}\}} \\
&= \mathrm{length}_R(R/(\mathfrak{q} + G_R(M)))[R/\mathfrak{m}].
\end{aligned}$$

6.4 Concluding remarks

In this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$ and quotient field $k = R/\mathfrak{m}$. Furthermore, M and N denote finitely generated modules with $\operatorname{pd}_R M < \infty$ and $\dim_R(M \otimes_R N) = 0$.

In section 6.2 we proved the intersection conjectures in the case that $\dim_R N = 0$ by working in the Grothendieck group $G_0^R(\text{f,P})$ in which we could exploit the identity $[M] = \chi^R(M)[R]$, whereby we proved that

$$\chi^R(M, N) = \chi^R(M) \operatorname{length}_R N.$$

The vanishing and positivity conjectures then followed from the bi-implication $\dim_R M = \dim R \iff \chi^R(M) \neq 0$. Similarly, we proved the intersection conjectures in the case $\dim_R N = 1$ by working in the Grothendieck group $G_0^R(\text{f,P}|S\text{-tor})$ for a multiplicative system S of R, in which we could exploit the identity $[M] = [R/G_R(M)]$, whereby we proved, assuming that $N = R/\mathfrak{q}$ for a prime ideal $\mathfrak{q}$, that

$$\chi^R(M, N) = \operatorname{length}_R R/(G_R(M) + \mathfrak{q}).$$

The vanishing and positivity conjecture then followed from the bi-implication $\dim_R M = \dim R - 1 \iff G_R(M) \neq R$.

It is tempting to hope that these proofs can be generalized to higher dimensions of N, although the counterexample by Dutta, Hochster and McLaughlin shows that we cannot hope for a generalization to *all* dimensions of N without assuming that $\operatorname{pd}_R N < \infty$. A natural place to start is the case that $\dim_R N = 2$. Since we have already dealt with the cases that $\operatorname{grade}_R M = 0, 1$, we may assume that $\operatorname{grade}_R M \geq 2$. It immediately follows that

$$\dim_R N + \dim_R M \leq \dim_R N + \dim R - \operatorname{grade}_R M \leq \dim R,$$

so (0) of the intersection conjectures holds.

In general, one can imagine a situation where the intersection conjectures have been verified for $\dim_R N < n$ and $\operatorname{grade}_R M < n$ for some $n > 1$. Considering the case $\dim_R N = n$, we can then assume that $\operatorname{grade}_R M \geq n$, and an argument similar to the one above then shows that (0) of the intersection conjectures holds. Moving on to vanishing and positivity along the lines of section 6.2, the first question that needs to be asked is which Grothendieck group $\chi^R(-, N)$ can be factored through. The answer is as follows.

Let $\mathfrak{N}_1$ denote the (finite) set of minimal primes in $\operatorname{Supp}_R N$. Using prime avoidance (Lemma 0.1), we can choose an element

$$x_1 \in \operatorname{Ann}_R M \backslash \big(\operatorname{Zd} R \cup \bigcup_{\mathfrak{q} \in \mathfrak{N}_1} \mathfrak{q} \big),$$

for otherwise $\operatorname{Ann}_R M$ would be contained in $\operatorname{Zd} R$, which is definitely not the case since $\operatorname{grade} M \geq n$, or in an prime $\mathfrak{q} \in \mathfrak{N}_1$, contradicting the assumptions that $\dim_R N = n$ and $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \{\mathfrak{m}\}$. We now have from Proposition 0.19(iv) that $\operatorname{grade}_R(\operatorname{Ann}_R M, R/\langle x_1\rangle) = \operatorname{grade}_R M - 1 \geq n-1$, and by choice of x_1 we have $\dim_R N/x_1N = \dim_R N - 1 = n-1$.

Letting $\mathfrak{N}_2$ denote the (finite) set of minimal elements in $\operatorname{Supp}_R N/x_1N$, we can next choose, using prime avoidance again, an element

$$x_2 \in \operatorname{Ann}_R M \setminus \big(\operatorname{Zd}_R(R/\langle x_1\rangle) \cup \bigcup_{\mathfrak{q}\in\mathfrak{N}_2} \mathfrak{q}\big),$$

for otherwise $\operatorname{Ann}_R M$ would be contained in $\operatorname{Zd}_R(R/\langle x_1\rangle)$ or in a prime from $\mathfrak{N}_2$, both situations contradicting the facts that $\operatorname{grade}_R(\operatorname{Ann}_R M, R/\langle x_1\rangle) \geq n-1$ and $\dim_R N/x_1N = n-1$. Continuing this process, we choose at the i'th step an element

$$x_i \in \operatorname{Ann}_R M \setminus \big(\operatorname{Zd}_R(R/\langle x_1,\ldots,x_{i-1}\rangle) \cup \bigcup_{\mathfrak{q}\in\mathfrak{N}_i} \mathfrak{q}\big),$$

where $\mathfrak{N}_i$ is the (finite) set of minimal elements in $\operatorname{Supp}_R(N/\langle x_1,\ldots,x_{i-1}\rangle N)$, and where we know that $\operatorname{grade}_R(\operatorname{Ann}_R M, R/\langle x_1,\ldots,x_{i-1}) \geq n-i+1$ and $\dim_R(N/\langle x_1,\ldots,x_{i-1}\rangle N) = n-i+1$.

The process terminates at the n'th step, where we have obtained a sequence $x = (x_1,\ldots,x_n)$ of elements in $\operatorname{Ann}_R M$ such that $\operatorname{grade}_R(\operatorname{Ann}_R M, R/\langle x\rangle) \geq 0$ and $\dim_R N/\langle x\rangle N = 0$. By construction, x is a regular sequence, and apparently x constitutes a system of parameters for N. Thus, we can factor $\chi^R(-,N)$ through the Grothendieck group $G_0^R(\text{f,P}|S(x)\text{-tor})$: from the way we constructed x, M clearly belongs to this Grothendieck group, and $\chi^R(-,N)$ is defined on any module from it. In attempting to calculate the intersection multiplicity $\chi^R(M,N)$, one can now replace M by any module that is identical to M within $G_0^R(\text{f,pd},S(x)\text{-tor})$. Following the lines of theorems 6.2 and 6.4, one could hope to replace M by a module that is sufficiently simple that we can easily calculate its intersection multiplicity with N. According to Corollary 3.38, we know that $[M]$ can be written as the difference of two elements represented by modules of projective dimension n.

This, however, is as far as our theory takes us! To get any further, one could look for an invariant to join the Euler characteristic and the MacRae ideal, which provided the necessary simplifications in dimensions 0 and 1. The theory of local Chern characters might assist in this search.

Proposition 5.5 showed that if $X \in \mathcal{C}_\square^R(\text{f,P})$ is a projective resolution of M, the zeroth local Chern character $\operatorname{ch}_0^R(X)$ is given as multiplication by $\chi^R(M)$, and Proposition 5.8 showed that the first local Chern character $\operatorname{ch}_1^R(X)$ is given by intersection with $G_R(M)$. Thus, the local Chern characters are definitely relevant in the search for an invariant to join the Euler characteristic and the MacRae ideal; one could imagine that such an invariant is related to the n'th local Chern character $\operatorname{ch}_n^R(X)$.

The connection between local Chern characters and the Euler characteristic and MacRae ideal is additionally emphasized in Corollaries 5.6 and 5.9. Together with Theorem 0.8, Corollary 5.6 shows that the following are equivalent for a module $M \in \mathcal{C}_0^R(\text{f,pd})$ with projective resolution $X \in \mathcal{C}_\square^R(\text{f,P})$.

(i) $\dim_R M = \dim R$.

(ii) $\operatorname{grade}_R M = 0$.

(iii) $\chi^R(M) \neq 0$.

(iv) $\operatorname{ch}_0^R(X) \neq 0$.

Assuming that none of these conditions are satisfied, Corollary 5.9 together with Proposition 4.23 show that the following are equivalent.

(i) $\dim_R M = \dim R - 1$.

(ii) $\operatorname{grade}_R M = 1$.

(iii) $G_R(M) \neq R$.

(iv) $\operatorname{ch}_1^R(X) \neq 0$.

It is thinkable that one could continue with four more conditions that are equivalent under the assumption that none of the above conditions are satisfied. Not only would this provide a valuable clue for how to attack the intersection conjectures in higher dimensions, it would also verify the grade conjecture in one more case. This is left for future studies.

Bibliography

[Bas74] Hyman Bass. *Introduction to some methods of algebraic K-theory.* American Mathematical Society, Providence, R.I., 1974. Expository Lectures from the CBMS Regional Conference held at Colorado State University, Ft. Collins, Colo., August 24-28, 1973, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 20.

[Ber01] George M. Bergman. *An invitation to general algebra and universal constructions.* 2.2 edition, August 2001. Lecture notes for the graduate course MATH250A at the University of California at Berkeley.

[BH93] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1993.

[Eis95] David Eisenbud. *Commutative algebra. With a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[Fox81] Hans-Bjørn Foxby. Intersection multiplicities of modules. Unpublished preprint, May 1981.

[Fox82a] Hans-Bjørn Foxby. K-theory for complexes with finite length homology. Copenhagen University Preprint Series no. 1, 1982.

[Fox82b] Hans-Bjørn Foxby. The Macrae invariant. In *Commutative algebra: Durham 1981 (Durham, 1981)*, volume 72 of *London Math. Soc. Lecture Note Ser.*, pages 121–128. Cambridge Univ. Press, Cambridge, 1982.

[Fox89] Hans-Bjørn Foxby. Notes on local Chern characters. Personal notes for a lecture held in Barcelona, 1989.

[Fox98] Hans-Bjørn Foxby. Hyperhomological Algebra & Commutative Rings. Unpublished notes, September 1998.

[Fox99] Hans-Bjørn Foxby. Homologisk algebra *[Homological Algebra]*. Lecture notes for the graduate course MATEMATIK 4HA at the University of Copenhagen, 1997 (with an addition from 1999).

[Hal03] Esben Bistrup Halvorsen. Betti numbers and Euler characteristics. Can be downloaded from `http://www.math.ku.dk/~ebh/papers.html`. Course project related to the graduate course MATH253 at the University of California at Berkeley, January 2003.

[HS97] Peter J. Hilton and Urs Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[Lan93] Serge Lang. *Algebra*. Addison-Wesley Publishing Company, Reading, MA, third edition, 1993.

[Mac65] Robert E. MacRae. On an application of the Fitting invariants. *J. Algebra*, 2:153–169, 1965.

[Mag02] Bruce A. Magurn. *An algebraic introduction to K-theory*, volume 87 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.

[Rob85] Paul C. Roberts. The vanishing of intersection multiplicities of perfect complexes. *Bull. Amer. Math. Soc. (N.S.)*, 13(2):127–130, 1985.

[Rob87] Paul C. Roberts. The MacRae invariant and the first local Chern character. *Trans. Amer. Math. Soc.*, 300(2):583–591, 1987.

[Rob92] Paul C. Roberts. The homological conjectures. In *Free resolutions in commutative algebra and algebraic geometry (Sundance, UT, 1990)*, volume 2 of *Res. Notes Math.*, pages 121–132. Jones and Bartlett, Boston, MA, 1992.

[Rob98] Paul C. Roberts. *Multiplicities and Chern classes in local algebra*, volume 133 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998.

[Rob03] Paul C. Roberts. Cycles and commutative algebra. Can be downloaded from `http://arxiv.org/abs/math.AC/0310448/`. E-print published on arXiv, 2003.

[Ser65] Jean-Pierre Serre. *Algèbre locale. Multiplicités [Local algebra. Multiplicities]*, volume 11 of *Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Troisième édition, 1975. Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1965.