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Intersection multiplicities and Grothendieck spaces

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- $\mathcal{P}_f(R) =$ the full subcategory of finite complexes isomorphic to a bounded complex of projectives.
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Throughout, $R$ denotes a commutative, Noetherian, local ring with maximal ideal $m$. Define

- $D(R) =$ the derived category of $R$.
- $D^f_\square(R) =$ the full subcategory of finite complexes.
- $P^f(R) =$ the full subcategory of finite complexes isomorphic to a bounded complex of projectives.

Let $X, Y \in D^f_\square(R)$ with $\text{Supp } X \cap \text{Supp } Y = \{m\}$. The intersection multiplicity of $X$ and $Y$ is defined as

$$\chi(X, Y) = \sum_i (-1)^i \text{length } H_i(X \otimes_R^L Y)$$

whenever $X \in P^f(R)$ or $Y \in P^f(R)$.
The ring $R$ satisfies vanishing if

$$\chi(X, Y) = 0 \text{ when } \dim(\text{Supp } X) + \dim(\text{Supp } Y) < \dim R.$$
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INTERSECTION MULTIPLICITIES

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**Theorem** (Dutta–Hochster–McLaughlin). *The vanishing conjecture does not hold, so not all rings satisfy vanishing!*
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**Theorem** (Dutta–Hochster–McLaughlin). *The vanishing conjecture does not hold, so not all rings satisfy vanishing!*

So far, all rings satisfy weak vanishing.
Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field.
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$\text{Supp } X \cap \text{Supp } Y = \{m\}$, $X \in P^f(R)$ and 
$\dim(\text{Supp } X) + \dim(\text{Supp } Y) \leq \dim R$. 
Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $X, Y \in D^f_{\mathfrak{m}}(R)$ with $\text{Supp } X \cap \text{Supp } Y = \{m\}$, $X \in \text{P}^f(R)$ and $\dim(\text{Supp } X) + \dim(\text{Supp } Y) \leq \dim R$. The Dutta multiplicity of $X$ and $Y$ is defined as

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\chi_{\infty}(X,Y) = \lim_{e \to \infty} \frac{1}{p^e \text{codim}(\text{Supp } X)} \chi(LF^e(X), Y).
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Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $X, Y \in \mathcal{D}^f_\square(R)$ with $\text{Supp } X \cap \text{Supp } Y = \{m\}$, $X \in \mathcal{P}^f(R)$ and $\text{dim}(\text{Supp } X) + \text{dim}(\text{Supp } Y) \leq \text{dim } R$. The Dutta multiplicity of $X$ and $Y$ is defined as

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where $\mathcal{L}F$ is the left-derived Frobenius functor.
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X = \cdots \longrightarrow X_i \overset{\partial^X_i}{\longrightarrow} X_{i-1} \longrightarrow \cdots
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Let $C$ be a full subcategory of the category of finite complexes.
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$$[X] = 0$$

whenever $X$ is exact

$$[X] = [X'] + [X'']$$

whenever $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ is a short exact sequence in $C$. 
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Since $\chi(-, Y)$ is zero on exact complexes and additive on short exact sequences, it factors through a Grothendieck group.
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Grothendieck Groups

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where \([LF^e(X)] \in K_0(C)\).
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where \( p^{-e \operatorname{codim}(\text{Supp} X)} [LF^e(X)] \in K_0(C) \otimes_{\mathbb{Z}} \mathbb{Q} \).
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where ???
Let $\mathcal{X}$ be a specialization-closed subset of $\text{Spec } R$. 
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$$p \in \mathcal{X} \quad \text{and} \quad p \subseteq q \quad \text{implies} \quad q \in \mathcal{X}. $$
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Define

$$D^{f}_{\square}(\mathcal{X}) = \text{the full subcategory of } D^{f}_{\square}(R) \text{ of complexes with support contained in } \mathcal{X}.$$
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Let $\mathcal{X}^c$ denote the maximal subset of $\text{Spec } R$ such that

$$\mathcal{X} \cap \mathcal{X}^c = \{m\} \quad \text{and} \quad \dim \mathcal{X} + \dim \mathcal{X}^c \leq \dim R.$$
Let $\mathcal{X}$ be a specialization-closed subset of $\text{Spec } R$. 
Let $\mathcal{X}$ be a specialization-closed subset of $\text{Spec } R$. The Grothendieck space of $\mathcal{P}^f(\mathcal{X})$ is the $\mathbb{Q}$-vector space $G\mathcal{P}^f(\mathcal{X})$ presented by generators $[X]$, one for each isomorphism class of a complex $X$ in $\mathcal{P}^f(\mathcal{X})$, and relations

$$[X] = [X'] \text{ when } \chi(X, -) = \chi(X', -)$$

as metafunctions $D^f_{\square}(\mathcal{X}^c) \to \mathbb{Q}$. 
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as metafunctions $D^f_{\Box}(\mathcal{X}^c) \to \mathbb{Q}$.

The space $\mathcal{G}\mathcal{P}^f(\mathcal{X})$ is equipped with the initial topology of the family of $\mathbb{Q}$-linear maps

$$\chi(-, Y): \mathcal{G}\mathcal{P}^f(\mathcal{X}) \to \mathbb{Q} \quad \text{for} \quad Y \in D^f_{\Box}(\mathcal{X}^c).$$
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where \( \mathfrak{X} = \text{Supp } X \).
Now we can calculate

\[ \chi_\infty(X, Y) = \lim_{e \to \infty} \frac{1}{pe \cdot \text{codim } \mathfrak{X}} \chi([LF^e(X)], Y) \]

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\[ = \lim_{e \to \infty} \chi\left(\frac{1}{p^e \text{codim } \mathcal{X}} F^e_{\mathcal{X}}([X]), Y \right) \]

where \( \mathcal{X} = \text{Supp } X \) and \( F^e_{\mathcal{X}}([X]) = [LF^e(X)] \).
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= \lim_{e \to \infty} \chi(\frac{1}{p^e \text{codim } \mathcal{X}}F^e_{\mathcal{X}}([X]), Y)
\]

\[
= \lim_{e \to \infty} \chi(\Phi^e_{\mathcal{X}}([X]), Y)
\]

where \( \mathcal{X} = \text{Supp } X \) and \( \Phi^e_{\mathcal{X}} = p^{-e \text{codim } \mathcal{X}}F^e_{\mathcal{X}} \).
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where $\mathcal{X} = \text{Supp } X$ and $\lim_{e \to \infty} \Phi^e\mathcal{X}([X]) \in \mathcal{G}P^f(\mathcal{X})$. 
Theorem 1. Let $\mathcal{X}$ be a specialization-closed subset of $\text{Spec } R$ and let $\alpha \in \mathcal{G}^{f}(\mathcal{X})$. 
**Theorem 1.** Let $\mathfrak{X}$ be a specialization-closed subset of $\text{Spec } R$ and let $\alpha \in \mathcal{G} \mathcal{P}^f(\mathfrak{X})$. Then there is a unique decomposition

$$\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$$

in which each $\alpha^{(i)}$ is either zero or an eigenvector for $\Phi_\mathfrak{X}$ with eigenvalue $p^{-i}$. 
Theorem 1. Let $\mathfrak{X}$ be a specialization-closed subset of $\text{Spec } R$ and let $\alpha \in \mathsf{GP}_f(\mathfrak{X})$. Then there is a unique decomposition

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in which each $\alpha^{(i)}$ is either zero or an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $p^{-i}$. The components $\alpha^{(0)}, \ldots, \alpha^{(u)}$ are recursively defined by

$$\alpha^{(0)} = \lim_{e \to \infty} \Phi_{\mathfrak{X}}^e(\alpha) \quad \text{and}$$

$$\alpha^{(i)} = \lim_{e \to \infty} p^{ie} \Phi_{\mathfrak{X}}^e(\alpha - (\alpha^{(0)} + \cdots + \alpha^{(i-1)})),$$
...and there is a formula

\[
\begin{pmatrix}
\alpha^{(0)} \\
\vdots \\
\alpha^{(u)}
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & p^{-1} & \cdots & p^{-u} \\
\vdots & \vdots & \ddots & \vdots \\
1 & p^{-u} & \cdots & p^{-u^2}
\end{pmatrix}^{-1}
\begin{pmatrix}
\alpha \\
\Phi \chi(\alpha) \\
\vdots \\
\Phi_u \chi(\alpha)
\end{pmatrix}.
\]
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\end{pmatrix}^{-1}
\begin{pmatrix}
\alpha \\
\Phi_x(\alpha) \\
\vdots \\
\Phi_x^u(\alpha)
\end{pmatrix}.
\]

The number \( u \) is the **vanishing dimension** of \( \alpha \); it measures, in a sense, how far \( \alpha \) is from satisfying vanishing. In particular, \( \alpha \) satisfies vanishing if and only if \( \alpha = \alpha^{(0)} \).
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We have \( u \leq \max(\text{codim } \mathcal{X} - 2, 0) \).
Translating the theorem to complexes, the Dutta multiplicity $\chi_\infty(X, Y)$ can be computed.
Translating the theorem to complexes, the Dutta multiplicity \( \chi_\infty(X, Y) \) is the first entry in

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
p^t & p^{t-1} & \ldots & p^{t-u} \\
\vdots & \vdots & \ddots & \vdots \\
p^{u(t)} & p^{u(t-1)} & \ldots & p^{u(t-u)}
\end{pmatrix}
\begin{pmatrix}
\chi(X, Y) \\
\chi(LF(X), Y) \\
\vdots \\
\chi(LF^u(X), Y)
\end{pmatrix}
\]

where \( t = \text{codim}(\text{Supp } X) \).
NUMERICAL PROPERTIES

Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field.
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Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $\mathcal{X}$ be a specialization-closed subset of $\text{Spec } R$ and let $\alpha \in \mathbb{G}P^f(\mathcal{X})$. If $\chi(\alpha, Y) = \chi(\alpha^{(0)}, Y)$ for all $Y \in \mathbb{P}^f(\mathcal{X}^c)$, then $\alpha$ satisfies numerical vanishing.
Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $\mathcal{X}$ be a specialization-closed subset of $\text{Spec } R$ and let $\alpha \in \text{GP}^f(\mathcal{X})$. If $\chi(\alpha, Y) = \chi(\alpha^{(0)}, Y)$ for all $Y \in \text{P}^f(\mathcal{X}^c)$, then $\alpha$ satisfies numerical vanishing. If this holds for all elements in all Grothendieck spaces, then $R$ satisfies numerical vanishing.
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$\chi(\alpha, Y) = \chi(\alpha^{(0)}, Y)$ for all $Y \in \mathbb{P}^f(\mathcal{X}^c)$, then $\alpha$ satisfies numerical vanishing. If this holds for all elements in all Grothendieck spaces, then $R$ satisfies numerical vanishing.

**Theorem 2.** The ring $R$ satisfies numerical vanishing if and only if all elements of $\mathbb{G}^f(\{m\})$ do.
Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $\mathcal{X}$ be a specialization-closed subset of $\text{Spec} \ R$ and let $\alpha \in \text{GP}^f(\mathcal{X})$. If $\chi(\alpha, Y) = \chi(\alpha^{(0)}, Y)$ for all $Y \in \text{P}^f(\mathcal{X}^c)$, then $\alpha$ satisfies numerical vanishing. If this holds for all elements in all Grothendieck spaces, then $R$ satisfies numerical vanishing.

**Theorem 2.** The ring $R$ satisfies numerical vanishing if and only if all elements of $\text{GP}^f(\{m\})$ do. In particular, this holds if and only if

$$\chi(\text{LF}(Z)) = p^{\dim R} \chi(Z)$$

for all complexes $Z$ in $\text{P}^f(\{m\})$. 
The duality functor $\mathbf{R} \text{Hom}_R(-, R)$ on $\mathcal{P}^f(\mathcal{X})$ induces an automorphism $(-)^*$ on $\mathcal{G} \mathcal{P}^f(\mathcal{X})$. 
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An element $\alpha \in \mathcal{GP}^f(\mathcal{X})$ is self-dual if it satisfies $\alpha = (-1)^{\text{codim} \mathcal{X}} \alpha^*$.
The duality functor $\mathbf{RHom}_R(-, R)$ on $\mathcal{P}^f(\mathcal{X})$ induces an automorphism $(-)^*$ on $\mathcal{G}^f(\mathcal{X})$.

An element $\alpha \in \mathcal{G}^f(\mathcal{X})$ is self-dual if it satisfies
\[\alpha = (-1)^{\text{codim } \mathcal{X}} \alpha^*,\]
and $\alpha$ is \textbf{numerically self-dual} if it satisfies
\[\chi(\alpha, Y) = (-1)^{\text{codim } \mathcal{X}} \chi(\alpha^*, Y)\]
for all $Y \in \mathcal{P}^f(\mathcal{X}^c)$.
The duality functor $\mathbf{R} \text{Hom}_R(-, R)$ on $\mathcal{P}^f(\mathcal{X})$ induces an automorphism $(-)^*$ on $\mathcal{GP}^f(\mathcal{X})$.

An element $\alpha \in \mathcal{GP}^f(\mathcal{X})$ is self-dual if it satisfies $\alpha = (-1)^{\text{codim} \mathcal{X}} \alpha^*$, and $\alpha$ is numerically self-dual if it satisfies $\chi(\alpha, Y) = (-1)^{\text{codim} \mathcal{X}} \chi(\alpha^*, Y)$ for all $Y \in \mathcal{P}^f(\mathcal{X}^c)$. If this holds for all elements in all Grothendieck spaces, then $R$ satisfies self-duality or numerical self-duality, respectively.
The duality functor $\mathbf{RHom}_R(\cdot, R)$ on $\mathbb{P}^f(\mathcal{X})$ induces an automorphism $(-)^*$ on $\mathbb{GP}^f(\mathcal{X})$.

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**Theorem 3** (w. Frankild). *When $R$ is complete of prime characteristic $p$ and with perfect residue field,*

$$(-1)^{\text{codim} \mathcal{X}} \alpha^* = \alpha^{(0)} - \alpha^{(1)} + \cdots + (-1)^u \alpha^{(u)}.$$
vanishing

weak vanishing
vanishing

weak vanishing
self-duality

vanishing

weak vanishing
self-duality

\[ \iff \]

vanishing

\[ \iff \]

numerical self-duality

\[ \iff \]

weak vanishing
Ring properties

self-duality

regular \rightarrow vanishing

numerical self-duality

weak vanishing
RING PROPERTIES

self-duality

regular \rightarrow \text{vanishing} \leftarrow \text{dim } \leq 2

numerical self-duality

weak vanishing
RING PROPERTIES

self-duality

\[ \iff \]

regular \[ \implies \] vanishing \[ \iff \] dim \( \leq 2 \)

\[ \implies \]

complete intersection

\[ \downarrow \]

numerical self-duality

\[ \downarrow \]

weak vanishing
RING PROPERTIES

self-duality

regular \rightarrow vanishing \leftarrow \text{dim} \leq 2

complete intersection

numerical self-duality

weak vanishing
RING PROPERTIES

self-duality

regular \rightarrow \text{vanishing} \leftarrow \text{dim} \leq 2

\text{complete intersection}

\text{numerical self-duality}

\text{weak vanishing} \leftarrow \text{dim} \leq 4
RING PROPERTIES

self-duality

regular \rightarrow \text{vanishing} \leftarrow \text{dim} \leq 2

\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow

complete intersection

numerical self-duality

\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow

Gorenstein of dim \leq 5

\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow

weak vanishing \leftrightarrow \text{dim} \leq 4
RING PROPERTIES

self-duality

regular

vanishing

dim \leq 2

complete intersection

numerical vanishing

numerical self-duality

Gorenstein of dim \leq 5

weak vanishing

dim \leq 4
RING PROPERTIES

self-duality

regular \rightarrow \text{vanishing} \leftarrow \text{dim} \leq 2

complete intersection \rightarrow \text{numerical vanishing}

numerical self-duality \leftarrow \text{Gorenstein of dim} \leq 5

weak vanishing \leftarrow \text{dim} \leq 4
Ring properties

self-duality

regular \quad \Rightarrow \quad \text{vanishing} \quad \iff \quad \text{dim} \leq 2

complete intersection \quad \Rightarrow \quad \text{numerical vanishing} \quad \iff \quad \text{Gorenstein of dim} \leq 3

\quad \Rightarrow \quad \text{numerical self-duality} \quad \iff \quad \text{Gorenstein of dim} \leq 5

\quad \Rightarrow \quad \text{weak vanishing} \quad \iff \quad \text{dim} \leq 4
RING PROPERTIES

self-duality

regular \rightarrow \text{vanishing} \iff \dim \leq 2

complete intersection \rightarrow \text{numerical vanishing} \leftarrow \text{Gorenstein of dim} \leq 3

Gorenstein \rightarrow \text{numerical self-duality} \leftarrow \text{Gorenstein of dim} \leq 5

Gorenstein \rightarrow \text{weak vanishing} \iff \dim \leq 4
RING PROPERTIES

self-duality

regular \rightarrow vanishing \leftrightarrow \text{dim } \leq 2

complete intersection \rightarrow numerical vanishing \leftrightarrow \text{dim } \leq 3

Gorenstein \rightarrow numerical self-duality \leftrightarrow \text{Gorenstein of dim } \leq 5

Cohen–Macaulay \rightarrow \text{weak vanishing } \leftrightarrow \text{dim } \leq 4
RING PROPERTIES

self-duality

regular \iff vanishing \iff \text{dim} \leq 2

complete intersection \iff numerical vanishing \iff Gorenstein of \text{dim} \leq 3

Gorenstein = = = = = \iff numerical self-duality \iff Gorenstein of \text{dim} \leq 5

Cohen–Macaulay \iff weak vanishing \iff \text{dim} \leq 4