GROTHENDIECK GROUPS FOR CATEGORIES OF COMPLEXES

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Abstract. The new intersection theorem states that, over a Noetherian local ring R, for any non-exact complex concentrated in degrees $n, \dots, 0$ in the category $\mathsf{P}(\text{length})$ of bounded complexes of finitely generated projective modules with finite length homology, we must have $n \geq d = \dim R$.

One of the results in this paper is that the Grothendieck group of $\mathsf{P}(\text{length})$ in fact is generated by complexes concentrated in the minimal number of degrees: if $\mathsf{P}_d(\text{length})$ denotes the full subcategory of $\mathsf{P}(\text{length})$ consisting of complexes concentrated in degrees $d, \dots, 0$, the inclusion $\mathsf{P}_d(\text{length}) \to \mathsf{P}(\text{length})$ induces an isomorphism of Grothendieck groups. When R is Cohen–Macaulay, the Grothendieck groups of $\mathsf{P}_d(\text{length})$ and $\mathsf{P}(\text{length})$ are naturally isomorphic to the Grothendieck group of the category $\mathsf{M}(\text{length})$ of finitely generated modules of finite length and finite projective dimension. This and a family of similar results are established in this paper.

1. Introduction

In this paper, we will prove the existence of isomorphisms between Grothendieck groups of various related categories of complexes. The paper presents a family of results that can all be formulated in a similar way. This introduction discusses only one of the results (as did the abstract); the remaining results can be obtained by replacing the property of "having finite length" with other properties of modules—see the next section for further details.

Let R be a commutative, Noetherian, local ring of dimension d. Let $\mathsf{P}(\text{length})$ denote the category of bounded complexes of finitely generated projective R-modules and finite length homology, and let $\mathsf{P}_d(\text{length})$ denote the full subcategory of complexes concentrated in degrees $d, \dots, 0$. We shall denote the Grothendieck groups of these two categories by $K_0\mathsf{P}(\text{length})$ and $K_0\mathsf{P}_d(\text{length})$, respectively. The inclusion of categories $\mathsf{P}_d(\text{length}) \to \mathsf{P}(\text{length})$ naturally induces a homomorphism

$$\mathcal{I}_d\colon K_0\mathsf{P}_d(\text{length}) \to K_0\mathsf{P}(\text{length}),$$

given by $\mathcal{I}_d([X]) = [X]$ for a complex $X \in \mathsf{P}_d(\text{length})$; here, the two $[X]$'s are different, since one is an element of $K_0\mathsf{P}_d(\text{length})$ and the other is an element of $K_0\mathsf{P}(\text{length})$. One of the results of this paper (Corollary 6) is that the above is an isomorphism. This is particularly interesting when comparing with the *new intersection theorem* (cf. [6, Theorem 13.4.1]), which states that, if a complex in $\mathsf{P}(\text{length})$ is non-exact and concentrated in degrees $n, \dots, 0$, then $n \geq d$. Thus, the Grothendieck group $K_0\mathsf{P}(\text{length})$ is generated by complexes with the minimal possible amplitude.

Next let $\mathsf{M}(\text{length})$ denote the category of R-modules of finite length and finite projective dimension. We denote the Grothendieck group of $\mathsf{M}(\text{length})$ by $K_0\mathsf{M}(\text{length})$. Any module in $\mathsf{M}(\text{length})$ has a projective resolution in $\mathsf{P}(\text{length})$, and there is a natural homomorphism

$$\mathcal{R}\colon K_0\mathsf{M}(\text{length}) \to K_0\mathsf{P}(\text{length}),$$

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given by $\mathcal{R}([M]) = [X]$ for a module $M \in \mathsf{M}(\text{length})$ with projective resolution $X \in \mathsf{P}(\text{length})$.

Now, suppose further that R is Cohen-Macaulay. The acyclicity lemma by Peskine and Szpiro (see [6, Theorem 4.3.2]) implies that the complexes in $\mathsf{P}_d(\text{length})$ are acyclic: that is, they are projective resolutions of their zeroth homology module. Taking the homology of a complex induces a natural homomorphism

$$\mathcal{H}_d \colon K_0\mathsf{P}_d(\text{length}) \to K_0\mathsf{M}(\text{length}),$$

given by $\mathcal{H}_d([X]) = [\mathrm{H}_0(X)]$ for a complex $X \in \mathsf{P}_d(\text{length})$.

The three homomorphisms that we have introduced so far fit together in a commutative diagram:

$$\begin{array}{ccc} K_0\mathsf{P}_e(\text{length}) & \xrightarrow{\quad \mathcal{I}_d \quad} & K_0\mathsf{P}(\text{length}) \\ & \searrow^{\mathcal{H}_d} \quad \nearrow_{\mathcal{R}} & \\ & K_0\mathsf{M}(\text{length}) & \end{array}$$

Here, $\mathcal{H}_d$ is dotted to emphasize the fact that it required an extra assumption to be defined. The fact that $\mathcal{I}_d$ is an isomorphism yields that so are $\mathcal{R}$ and $\mathcal{H}_d$, whenever defined (Corollary 11).

When replacing the property of "having finite length" with other module properties, the same picture will emerge. The next section presents all the results of this paper in a general way—including the results mentioned in this introduction.

Historical note: This paper builds on the first author's incomplete preprint [2] whose results have been generalized and completely proven by the second author. The paper will become part of the second author's Ph.D. thesis. The results are generalizations of a result by Roberts and Srinivas [7, Proposition 2]

2. Grothendieck groups for categories of complexes

Notation. Throughout this paper, R denotes a non-trivial, unitary, commutative ring. All modules are, unless otherwise stated, assumed to be R-modules, and all complexes are, unless otherwise stated, assumed to be complexes of R-modules. Modules are considered to be complexes concentrated in degree zero.

Let d be a non-negative integer and let $S = (S_1, \ldots, S_d)$ be a family of multiplicative systems of R. A module M is said to be S_i*-torsion* if $S_i^{-1}M = 0$, and M is said to be S*-torsion* if it is S_i-torsion for $i = 1, \ldots, d$. The *grade* of M is the number

$$\operatorname{grade}_R M = \inf\{n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(M, R) \neq 0\}.$$

If $M = 0$, we set $\operatorname{grade}_R M = \infty$. When R is Noetherian and M is non-trivial and finitely generated, $\operatorname{grade}_R M$ is the maximal length of a regular sequence in $\operatorname{Ann}_R M$. M is said to be d*-perfect* if $M = 0$ or $\operatorname{grade}_R M = d = \operatorname{pd}_R M$.

We shall use the following abbreviations for properties of modules.

S-tor:	being S-torsion;
length:	having finite length;
$\mathrm{gr} \geq d$:	having grade larger than or equal to d; and
d-perf:	being d-perfect.

Let e be a non-negative integer, and let the symbol # denote any of the module properties above. We define the following categories.

M: the category of finitely generated modules of finite projective dimension;
P: the category of bounded complexes of finitely generated projective modules;
P_e: the full subcategory of P consisting of complexes concentrated in degrees $e, \ldots, 0$;
$\mathsf{M}(\#)$: the full subcategory of M consisting of modules satisfying #;
$\mathsf{P}(\#)$: the full subcategory of P consisting of complexes whose homology modules satisfy #;
$\mathsf{P}_e(\#)$: the intersection of P_e and $\mathsf{P}(\#)$.

So, for example, $\mathsf{P}_e(S\text{-tor})$ denotes the category of complexes concentrated in degrees $e, \ldots, 0$ with finitely generated projective modules and S-torsion homology modules. We will allow the symbol # to be "empty" so that $\mathsf{M}(\#)$, $\mathsf{P}(\#)$ and $\mathsf{P}_e(\#)$ also can denote M, P and P_e, respectively. Similarly, we shall occasionally write $\mathsf{P}_\star(\#)$, where the symbol $\star$ either denotes a non-negative integer e or is "empty", in which case we are back with the category $\mathsf{P}(\#)$.

The isomorphism classes of any of the categories $\mathsf{M}(\#)$ and $\mathsf{P}_\star(\#)$ form a set. We shall occasionally abuse notation and use $\mathsf{M}(\#)$ and $\mathsf{P}_\star(\#)$ to denote the sets of isomorphism classes of the corresponding categories.

Definition 1. The *Grothendieck group* of a category $\mathsf{M}(\#)$ is the Abelian group $K_0\mathsf{M}(\#)$ presented by generators $[M]$, one for each isomorphism class in $\mathsf{M}(\#)$, and relations

$$[M] = [L] + [N] \quad \text{whenever} \quad 0 \to L \to M \to N \to 0$$

is a short exact sequence in $\mathsf{M}(\#)$.

The *Grothendieck group* of a category $\mathsf{P}_\star(\#)$ is the Abelian group $K_0\mathsf{P}_\star(\#)$ presented by generators $[X]$, one for each isomorphism class in $\mathsf{P}_\star(\#)$, and relations

$$[X] = 0 \quad \text{whenever } X \text{ is exact,}$$

and

$$[Y] = [X] + [Z] \quad \text{whenever} \quad 0 \to X \to Y \to Z \to 0$$

is a short exact sequence in $\mathsf{P}_\star(\#)$.

So, for example, $K_0\mathsf{P}_e(S\text{-tor})$ denotes the Grothendieck group of the category $\mathsf{P}_e(S\text{-tor})$, whereas the usual *zeroth algebraic K-group of R* is the group $K_0(R) = K_0\mathsf{P}_0$: that is, the Grothendieck group of the category of P_0.

In the following three propositions, we list some useful properties of Grothendieck groups, which will be used throughout this paper. The properties can easily be verified and are stated without proof; for more details, the reader is referred to Halvorsen [4, page 8-10].

Proposition 2. *Any element in $K_0\mathsf{M}(\#)$ can be written in the form $[M]-[M']$ for modules $M, M' \in \mathsf{M}(\#)$, and any element in $K_0\mathsf{P}_\star(\#)$ can be written in the form $[X]-[X']$ for complexes $X, X' \in \mathsf{P}_\star(\#)$.* □

If X is a complex, it can be *shifted n degrees to the left*, thereby yielding the complex $\Sigma^n X$ with modules $(\Sigma^n X)_\ell = X_{\ell-n}$ and differentials $\partial_\ell^{\Sigma^n X} = (-1)^n \partial_{\ell-n}^X$. In the case that $n = 1$, the operator $\Sigma^1(-)$ is simply denoted by $\Sigma(-)$.

Proposition 3. *Suppose that X is a complex in $\mathsf{P}_\star(\#)$ such that $\Sigma^n X$ is in $\mathsf{P}_\star(\#)$. Then $[\Sigma^n X] = (-1)^n[X]$ in $K_0\mathsf{P}_\star(\#)$.* □

Proposition 4. *Suppose that $\phi\colon X \to Y$ is a quasi-isomorphism in $\mathsf{P}_\star(\#)$ such that ΣX is in $\mathsf{P}_\star(\#)$. Then $[X] = [Y]$ in $K_0\mathsf{P}_\star(\#)$.* □

Note that, since quasi-isomorphisms become identities in the Grothendieck group, we might as well have modelled the Grothendieck groups on derived categories rather than usual categories.

Since $\mathsf{P}_e(\#)$ is a subcategory of $\mathsf{P}(\#)$, the inclusion of categories induces a natural group homomorphism

$$\mathcal{I}_e\colon K_0\mathsf{P}_e(\#) \to K_0\mathsf{P}(\#)$$

given by $\mathcal{I}_e([X]) = [X]$. Note that the two $[X]$'s here are different: one is an element of $K_0\mathsf{P}_e(\#)$, whereas the other is an element of $K_0\mathsf{P}(\#)$. Note also that the fact that $\mathcal{I}_e$ is induced by an inclusion of the underlying categories does not mean that $\mathcal{I}_e$ is injective—it only ensures that $\mathcal{I}_e$ is well defined.

When R is Noetherian, a module in $\mathsf{M}(\#)$ always has a projective resolution in $\mathsf{P}(\#)$. It follows from Proposition 4 that different projective resolutions of the same module always represent the same element in the Grothendieck group $K_0\mathsf{P}(\#)$. Thus, we can, to each module $M \in \mathsf{M}(\#)$ with projective resolution $X \in \mathsf{P}(\#)$, associate the element $[X]$ in $K_0\mathsf{P}(\#)$. Since the modules in a short exact sequences have projective resolutions that fit together in a short exact sequence, this association induces a group homomorphism

$$\mathcal{R}\colon K_0\mathsf{M}(\#) \to K_0\mathsf{P}(\#)$$

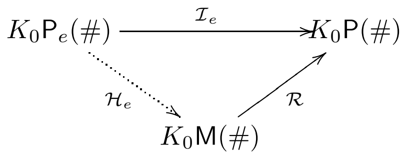
given by $\mathcal{R}([M]) = [X]$ where $X \in \mathsf{P}(\#)$ is a projective resolution of $M \in \mathsf{M}(\#)$.

As we shall see in the next section, certain additional assumptions on the ring together with a sufficiently small choice of e can force the homology of complexes in $\mathsf{P}_e(\#)$ to be concentrated in degree zero and hence be modules in $\mathsf{M}(\#)$. Thus, in this case, we can, to every complex $X \in \mathsf{P}_e(\#)$, associate the element $[\mathrm{H}(X)]$ in $K_0\mathsf{M}(\#)$, where H denotes the homology functor. Since this association clearly preserves the relations in $K_0\mathsf{P}_e(\#)$, it induces a group homomorphism

$$\mathcal{H}_e\colon K_0\mathsf{P}_e(\#) \to K_0\mathsf{M}(\#)$$

given by $\mathcal{H}_e([X]) = [\mathrm{H}(X)]$.

The homomorphisms $\mathcal{I}_e$, $\mathcal{H}_e$ and $\mathcal{R}$ are connected in a commutative diagram as shown below.



$\mathcal{H}_e$ is here dotted to underline the fact that it required an extra assumption to be defined. The homomorphism $\mathcal{R}$ always requires R to be Noetherian in order to be defined.

Let $x = (x_1, \dots, x_d)$ denote a regular sequence, and let $S(x)$ denote the family $(S(x_1), \dots, S(x_d))$ of multiplicative systems $S(x_i) = \{x_i^n \mid n \in \mathbb{N}_0\}$. Further, let T denote a (single) multiplicative system such that $T \cap \operatorname{Zd} R = \emptyset$. In the next section we shall prove that the homomorphisms $\mathcal{H}_e$ and $\mathcal{R}$ are defined under the assumptions on e and R described in the table below.

#	e	assumption on R
$S(x)$-tor	d	Noetherian, local
T-tor	1	Noetherian, local
$-$	0	Noetherian
length	$\dim R$	Noetherian, local, Cohen–Macaulay
$\operatorname{gr} \geq d$	d	Noetherian, local
d-perf	d	Noetherian, local

In this paper we will show that $\mathcal{I}_e$, $\mathcal{H}_e$ and $\mathcal{R}$ in all but the last of the above situations are isomorphisms and that, in the last situation, $\mathcal{I}_e$ and $\mathcal{R}$ are monomorphism and $\mathcal{H}_e$ is an isomorphism. These results will be derived as corollaries to the theorem below, which shall henceforth be referred to as the “Main Theorem”. As the proof of the Main Theorem will show, the Grothendieck group $K_0\mathsf{P}_e(\#)$, where $\#$ is any of the properties in the table above, is, in fact, isomorphic to $K_0\mathsf{P}(\#)$ whenever e is larger than or equal to the corresponding number in the table and trivial otherwise.

Main Theorem. *Suppose that d is a non-negative integer and that $S = (S_1, \dots, S_d)$ is a d-tuple of multiplicative systems of R. Then the homomorphism*

$$\mathcal{I}_d\colon K_0\mathsf{P}_d(S\text{-tor}) \to K_0\mathsf{P}(S\text{-tor})$$

given by $\mathcal{I}_d([X]) = [X]$ is an isomorphism.

Note that, in the setting of the Main Theorem, there are no additional requirements on R, and the homomorphisms $\mathcal{H}_d$ and $\mathcal{R}$ is not necessarily defined. However, when $\mathcal{H}_d$ and $\mathcal{R}$ are defined, we can immediately infer that $\mathcal{H}_d$ is injective and that $\mathcal{R}$ is surjective, and as it is not hard to see that $\mathcal{H}_d$ is surjective, it follows that all three homomorphisms are isomorphisms.

The Main Theorem says that any element of $K_0\mathsf{P}(S\text{-tor})$ can be represented by a linear combination of complexes concentrated in degrees $d, \dots, 0$. As we shall see, the inverse map $\mathcal{I}_d^{-1}\colon K_0\mathsf{P}(S\text{-tor}) \to K_0\mathsf{P}_d(S\text{-tor})$ is basically constructed from a procedure describing how to "make complexes smaller". When $\mathcal{H}_d$ is defined, the complexes become so small that they are forced to be resolutions of modules with projective dimension at most d.

When R is Noetherian and local, $d = 1$ and the multiplicative set T contains no zero-divisors, $\mathcal{H}_1\colon K_0\mathsf{P}_1(T\text{-tor}) \to K_0\mathsf{M}(T\text{-tor})$ is, as we shall see, defined and all of $\mathcal{I}_1$, $\mathcal{H}_1$ and $\mathcal{R}$ are isomorphisms. So in this case, the elements of $K_0\mathsf{M}(T\text{-tor})$ can be represented by elements in the form $[R^n/AR]$, where A is an injective $n \times n$-matrix. Using the localization sequence

$$K_1(R) \to K_1(T^{-1}R) \to K_0\mathsf{M}(T\text{-tor}) \to K_0(R) \to K_0(T^{-1}R)$$

of algebraic K-groups, it is not hard to see that $[R^n/AR^n] = [R/(\det A)R]$ in $K_0\mathsf{M}(T\text{-tor})$. Thus, $K_0\mathsf{P}_1(T\text{-tor})$ (and hence $K_0\mathsf{P}(T\text{-tor})$) is in fact generated by Koszul complexes. This property was fundamental in Foxby's proof in [3] of Serre's intersection conjectures in the case where one module has dimension 1.

The rather tedious proof of the Main Theorem is postponed until Section 4. For now, we will assume that it has been established and use it to derive all the other results.

3. Isomorphisms between Grothendieck groups

Definition 5. If x is an element of R, $S(x)$ denotes the multiplicative system $\{x^n \mid n \in \mathbb{N}_0\}$, and if $x = (x_1, \dots, x_d)$ is a d-tuple of elements from R, $S(x)$ denotes the d-tuple $(S(x_1), \dots, S(x_d))$ of multiplicative systems.

We begin our collection of corollaries to the Main Theorem with the result discussed in the abstract and the introduction.

Corollary 6. *If R is Noetherian and local with $\dim R = d$, then the group homomorphism $\mathcal{I}_d\colon K_0\mathsf{P}_d(\text{length}) \to K_0\mathsf{P}(\text{length})$ given by $\mathcal{I}_d([X]) = [X]$ is an isomorphism.*

Proof. Let $x = (x_1, \dots, x_d)$ be a system of parameters, and notice that a finitely generated module has finite length if and only if it is $S(x)$-torsion. Consequently, $K_0\mathsf{P}(\text{length}) = K_0\mathsf{P}(S(x)\text{-tor})$ and $K_0\mathsf{P}_d(\text{length}) = K_0\mathsf{P}_d(S(x)\text{-tor})$, and the result follows from the Main Theorem □

Lemma 7. *Suppose that R is Noetherian and let $x = (x_1, \dots, x_d)$ be a regular sequence of length $d > 0$. Then any complex X in $\mathsf{P}_d(S(x)\text{-tor})$ satisfies the condition* *that its homology complex* $\mathrm{H}(X)$ *is concentrated in degree* 0*: that is,* $\mathrm{H}(X)$ *is a module in* $\mathsf{M}(S(x)\text{-tor})$*.*

Proof. Let X be a non-exact complex in $\mathsf{P}_d(S(x)\text{-tor})$ and let t denote the largest integer such that $\mathrm{H}_t(X) \neq 0$; this exists since $\mathrm{H}(X) \neq 0$ and X is bounded. We already know that $t \geq 0$, so let us assume that $t > 0$ and try to reach a contradiction.

Let $\mathfrak{p}$ be an associated prime of $\mathrm{H}_t(X)$. Since $\mathrm{H}(X)$ is $S(x)$-torsion, we can find $N_1, \ldots, N_d \in \mathbb{N}$ such that $x_1^{N_1}, \ldots, x_d^{N_d} \in \operatorname{Ann}_R \mathrm{H}_t(X) \subseteq \mathfrak{p}$. Consequently, $(x_1/1, \ldots, x_d/1)$ is an $R_\mathfrak{p}$-sequence in $\mathfrak{p}_\mathfrak{p}$, so $\operatorname{depth} R_\mathfrak{p} \geq d \geq 1$.

Now, the projective resolution

$$0 \to (X_d)_\mathfrak{p} \to \cdots \to (X_{t+1})_\mathfrak{p} \to (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} \to 0$$

of $(\operatorname{im} \partial_{t+1}^X)_\mathfrak{p}$ as an $R_\mathfrak{p}$-module shows that $\operatorname{pd}_{R_\mathfrak{p}} (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} \leq d - (t+1)$. From the Auslander–Buchsbaum formula (see, for example, [1, Theorem 1.3.3]), it now follows that

$$\operatorname{depth}_{R_\mathfrak{p}} (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} - \operatorname{pd}_{R_\mathfrak{p}} (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} \geq t + 1 \geq 2.$$

Since $(\ker \partial_t^X)_\mathfrak{p}$ is a submodule of the non-trivial free $R_\mathfrak{p}$-module $(X_t)_\mathfrak{p}$ which has $\operatorname{depth}_{R_\mathfrak{p}} (X_t)_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} \geq d \geq 1$, we must also have $\operatorname{depth}_{R_\mathfrak{p}} (\ker \partial_t^X)_\mathfrak{p} \geq 1$. From the short exact sequence

$$0 \to (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} \to (\ker \partial_t^X)_\mathfrak{p} \to (\mathrm{H}_t(X))_\mathfrak{p} \to 0,$$

it now follows that $\operatorname{depth}_{R_\mathfrak{p}} (\mathrm{H}_t(X))_\mathfrak{p} \geq 1$ (see, for example, [1, Proposition 1.2.9]). This is a contradiction, however, because $\operatorname{depth}_{R_\mathfrak{p}} (\mathrm{H}_t(X))_\mathfrak{p} = 0$, since $\mathfrak{p}$ is associated to $\mathrm{H}_t(X)$. Thus, $t = 0$ as desired. □

Corollary 8. *If* R *is Noetherian and local, and* $x = (x_1, \ldots, x_d)$ *is a regular sequence of length* $d > 0$*, then there is a commutative diagram*

$$\begin{array}{ccc} K_0\mathsf{P}_d(S(x)\text{-tor}) & \xrightarrow{\quad \mathcal{I}_d \quad} & K_0\mathsf{P}(S(x)\text{-tor}) \\ & {\scriptstyle \mathcal{H}_d} \searrow \qquad \nearrow {\scriptstyle \mathcal{R}} & \\ & K_0\mathsf{M}(S(x)\text{-tor}) & \end{array}$$

in which $\mathcal{I}_d$*,* $\mathcal{H}_d$ *and* $\mathcal{R}$ *are isomorphisms.*

Proof. Lemma 7 shows that $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are well-defined homomorphisms, and the Main Theorem states that $\mathcal{I}_d$ is an isomorphism.Thus, we already know that $\mathcal{H}_d$ is injective and $\mathcal{R}$ is surjective.

Now, let M be a module in $\mathsf{M}(S(x)\text{-tor})$, and let us show by induction on $p = \operatorname{pd}_R M$ that $[M] \in \operatorname{im} \mathcal{H}_d$. If $p \leq d$, it is clear that $[M] \in \operatorname{im} \mathcal{H}$, since M in this case has a projective resolution in $\mathsf{P}_d(S(x)\text{-tor})$. So assume that $p > d$, and choose a finitely generated free module F and a surjective homomorphism $f\colon F \to M$. Next, using the fact that M is $S(x)$-torsion, choose $N_1, \ldots, N_d \in \mathbb{N}$ so that $x_1^{N_1}, \ldots, x_d^{N_d} \in \operatorname{Ann}_R M$, and let $\overline{F} = F/(x_1^{N_1}, \ldots, x_d^{N_d})F$. The surjection f induces a surjection $\overline{f}\colon \overline{F} \to M$. Letting K denote the kernel of $\overline{f}$, we then have an exact sequence

$$0 \to K \to \overline{F} \to M \to 0,$$

and since $\operatorname{pd}_R \overline{F} = d < p = \operatorname{pd}_R M$, it follows that $\operatorname{pd}_R K = d - 1$. By construction, $\overline{F}$ and K are $S(x)$-torsion, so $\overline{F}$ and K are modules in $\mathsf{M}(S(x)\text{-tor})$, and the induction hypothesis yields $[M] = [\overline{F}] - [K] \in \operatorname{im} \mathcal{H}_d$. Consequently $\mathcal{H}_d$ is surjective, and it follows that $\mathcal{H}_d$ as well as $\mathcal{R}$ are isomorphisms. □

Corollary 8 also holds in the case $d = 0$, where the requirement of being $S(x)$-torsion drops out, even without the assumption that R is local. We state this as a separate corollary and leave the straightforward proof to the reader.

Corollary 9. *If R is Noetherian, then there is a commutative diagram*

$$\begin{array}{ccc} K_0(R) = K_0\mathsf{P}_0 & \xrightarrow{\mathcal{I}_0} & K_0\mathsf{P} \\ & {\scriptstyle\mathcal{H}_0}\searrow \quad \nearrow{\scriptstyle\mathcal{R}} & \\ & K_0\mathsf{M} & \end{array}$$

in which $\mathcal{I}_0$, $\mathcal{H}_0$ and $\mathcal{R}$ are isomorphisms. □

When R in addition is local, the Grothendieck groups in Corollary 9 are all isomorphic to $\mathbb{Z}$ through the rank on $K_0\mathsf{P}_0$. As the proof of the Main Theorem (Theorem 42) will show, the isomorphism $K_0\mathsf{P} \to \mathbb{Z}$ is given by taking an element $[X] \in K_0\mathsf{P}$ to the integer $\sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{rank}_R X_\ell$, whereas the isomorphism $K_0\mathsf{M} \to \mathbb{Z}$ is given by taking and element $[M] \in K_0\mathsf{M}$ to the Euler characteristic $\chi^R(M)$, defined as the alternating sum of the ranks in a finite free resolution of M.

The proofs of Lemma 7 and Corollary 8 in the case $d = 1$ clearly show that the multiplicative system $S(x) = S(x_1) = \{x_1^n \mid n \in \mathbb{N}_0\}$ can be replaced by *any* multiplicative system S containing only non-zerodivisors. This is because any element of such a multiplicative system in itself constitutes a regular sequence of length 1. We state this as a separate corollary.

Corollary 10. *If R is Noetherian and T is a multiplicative system with $T \cap \operatorname{Zd} R = \emptyset$, then there is a commutative diagram*

$$\begin{array}{ccc} K_0\mathsf{P}_1(T\text{-tor}) & \xrightarrow{\mathcal{I}_1} & K_0\mathsf{P}(T\text{-tor}) \\ & {\scriptstyle\mathcal{H}_1}\searrow \quad \nearrow{\scriptstyle\mathcal{R}} & \\ & K_0\mathsf{M}(T\text{-tor}) & \end{array}$$

in which $\mathcal{I}_1$, $\mathcal{H}_1$ and $\mathcal{R}$ are isomorphisms. □

Another special case of Corollary 8 that we would like to point out is the case $d = \dim R$, which is only possible when R is Cohen–Macaulay. In this case, the property of being S-torsion is identical to the property of having finite length. The result in this case was discussed in the abstract and the introduction, and we also state it as a separate corollary.

Corollary 11. *If R is a Noetherian, local Cohen-Macaulay ring of dimension d, then there is a commutative diagram*

$$\begin{array}{ccc} K_0\mathsf{P}_d(\text{length}) & \xrightarrow{\mathcal{I}_d} & K_0\mathsf{P}(\text{length}) \\ & {\scriptstyle\mathcal{H}_d}\searrow \quad \nearrow{\scriptstyle\mathcal{R}} & \\ & K_0\mathsf{M}(\text{length}) & \end{array}$$

in which $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are isomorphisms. □

As we shall see in Corollary 12 below, Corollary 8 can also be used to derive results concerning the property of having grader larger than or equal to d.

Corollary 12. *If R is Noetherian and local, and d is a positive integer, then there is a commutative diagram*

$$\begin{array}{ccc} K_0\mathsf{P}_d(\text{gr} \geq d) & \xrightarrow{\mathcal{I}_d} & K_0\mathsf{P}(\text{gr} \geq d) \\ & {\scriptstyle\mathcal{H}_d}\searrow \quad \nearrow{\scriptstyle\mathcal{R}} & \\ & K_0\mathsf{M}(\text{gr} \geq d) & \end{array}$$

in which $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are isomorphisms.

Proof. If d is so large that there are no regular sequences in R of length d, then the involved Grothendieck groups are all trivial and the theorem holds. We can therefore assume that regular sequences of length d do exist.

If X is a complex in $\mathsf{P}_d(\mathrm{gr} \geq d)$, we can find a regular sequence $x = (x_1, \dots, x_d)$ of length d contained in the annihilator of *all* the homology modules of X. Then X will be homologically $S(x)$-torsion, and it follows from Lemma 7 that the homology of X is concentrated in degree 0. Consequently, $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are well-defined homomorphisms.

We define an equivalence relation on the set of regular sequences, letting a regular sequence $x = (x_1, \dots, x_d)$ be equivalent to a regular sequence $x' = (x'_1, \dots, x'_d)$ whenever

$$\mathrm{Rad}_R(x_1, \dots, x_d) = \mathrm{Rad}_R(x'_1, \dots, x'_d),$$

where the radical $\mathrm{Rad}_R I$ of an ideal I is the intersection of all prime ideals containing I. It is clear that this, indeed, is an equivalence relation. Denote the set of equivalence classes by E, and partially order E by reversed inclusion of radical ideals: that is,

$$x \preccurlyeq x' \overset{\text{def}}{\iff} \mathrm{Rad}_R(x_1, \dots, x_d) \supseteq \mathrm{Rad}_R(x'_1, \dots, x'_d)$$

for $x, x' \in E$. (It is of course the *equivalence classes* of x and x' that belong to E, but this unimportant technicality will be ignored here.) $E = (E, \preccurlyeq)$ is a directed set, for if x and x' are regular sequences of length d, then we can find a regular sequence x'' of length d contained in $(x) \cap (x')$ and hence satisfying the condition that $x, x' \preccurlyeq x''$.

Now, the category $\mathsf{M}(S(x)\text{-tor})$ is uniquely determined by the equivalence class of x in E, since, for any finitely generated module M,

$$\begin{aligned} M \text{ is } S(x)\text{-torsion} &\iff \forall \nu \in \{1, \dots, d\} \exists N_\nu \in \mathbb{N}_0 \colon x_\nu^{N_\nu} \in \mathrm{Ann}_R M \\ &\iff (x_1, \dots, x_d) \subseteq \mathrm{Rad}_R(\mathrm{Ann}_R M) \\ &\iff \mathrm{Rad}_R(x_1, \dots, x_d) \subseteq \mathrm{Rad}_R(\mathrm{Ann}_R M). \end{aligned}$$

Thus, we can consider the family of Grothendieck groups $K_0\mathsf{M}(S(x)\text{-tor})$ indexed by the equivalence classes in E. Given $x, x' \in E$ with $x \preccurlyeq x'$, there is a homomorphism

$$\mathcal{I}_{x,x'} \colon K_0\mathsf{M}(S(x)\text{-tor}) \to K_0\mathsf{M}(S(x')\text{-tor})$$

given by $\mathcal{I}_{x,x'}([M]) = [M]$; this is well defined, since it is induced by an inclusion of categories as seen from the bi-implications above. Consequently, $(K_0\mathsf{M}(S(x)\text{-tor}), \mathcal{I}_{x,x'})_{x \preccurlyeq x'}$ is a direct system, and it is straightforward to see that the Grothendieck group $K_0\mathsf{M}(\mathrm{gr} \geq d)$ together with the natural homomorphisms $\tau_x \colon K_0\mathsf{M}(S(x)\text{-tor}) \to K_0\mathsf{M}(\mathrm{gr} \geq d)$ induced by the inclusion of the underlying categories and given by $\tau_x([M]) = [M]$, $x \in E$, satisfy the universal property required by a direct limit of this system.

We have now shown that $K_0\mathsf{M}(\mathrm{gr} \geq d)$ is the direct limit of the direct system $(K_0\mathsf{M}(S(x)\text{-tor}), \mathcal{I}_{x,x'})_{x \preccurlyeq x'}$. By the same methods one can show that $K_0\mathsf{P}_d(\mathrm{gr} \geq d)$ and $K_0\mathsf{P}(\mathrm{gr} \geq d)$ are the direct limits of the direct systems $(K_0\mathsf{P}_d(S(x)\text{-tor}), \mathcal{I}_{x,x'})_{x \preccurlyeq x'}$ and $(K_0\mathsf{P}(S(x)\text{-tor}), \mathcal{I}_{x,x'})_{x \preccurlyeq x'}$, respectively, where the homomorphisms $\mathcal{I}_{x,x'}$ now are given by $\mathcal{I}_{x,x'}([X]) = [X]$ for complexes X in $\mathsf{P}_d(S(x)\text{-tor})$ and $\mathsf{P}(S(x)\text{-tor})$, respectively. Now, we already know from Corollary 8 that there is a commutative diagram of isomorphisms

$$\begin{array}{ccc} K_0\mathsf{P}_d(S(x)\text{-tor}) & \xrightarrow{\mathcal{I}_d} & K_0\mathsf{P}(S(x)\text{-tor}) \\ & {\scriptstyle\mathcal{H}_d} \searrow \quad \nearrow {\scriptstyle\mathcal{R}} & \\ & K_0\mathsf{M}(S(x)\text{-tor}) & \end{array}$$

for all $x \in E$, and hence there must also be a commutative diagram of isomorphisms

$$\begin{array}{ccccc} K_0\mathsf{P}_d(\mathrm{gr} \geq d) & & \xrightarrow{\quad \mathcal{I}_d \quad} & & K_0\mathsf{P}(\mathrm{gr} \geq d) \\ & {\scriptstyle \mathcal{H}_d}\searrow & & \nearrow{\scriptstyle \mathcal{R}} & \\ & & K_0\mathsf{M}(\mathrm{gr} \geq d) & & \end{array}$$

involving the direct limits. □

Because of Lemma 7, the homology of any complex in $\mathsf{P}_d(\mathrm{gr} \geq d)$ must be a d-perfect module. Thus, $\mathsf{P}_d(\mathrm{gr} \geq d) = \mathsf{P}_d(d\text{-perf})$, and hence $K_0\mathsf{P}_d(\mathrm{gr} \geq d) = K_0\mathsf{P}_d(d\text{-perf})$. It follows that the isomorphisms $\mathcal{H}_d \colon K_0\mathsf{P}_d(\mathrm{gr} \geq d) \to K_0\mathsf{M}(\mathrm{gr} \geq d)$ and $\mathcal{I}_d \colon K_0\mathsf{P}_d(\mathrm{gr} \geq d) \to K_0\mathsf{P}(\mathrm{gr} \geq d)$ from Corollary 12 must factor through $K_0\mathsf{M}(d\text{-perf})$ and $K_0\mathsf{P}(d\text{-perf})$, respectively. This is discussed in Corollary 13 below, which extends Corollary 12, and where we let $\tau \colon K_0\mathsf{M}(d\text{-perf}) \to K_0\mathsf{M}(\mathrm{gr} \geq d)$ and $\overline{\tau} \colon K_0\mathsf{P}(d\text{-perf}) \to K_0\mathsf{P}(\mathrm{gr} \geq d)$ denote the natural homomorphisms induced by the inclusion of the underlying categories and given by $\tau([M]) = [M]$ for $M \in \mathsf{M}(d\text{-perf})$ and $\overline{\tau}([X]) = [X]$ for $X \in \mathsf{P}(d\text{-perf})$.

Corollary 13. *If R is Noetherian and local and d is a positive integer, then there is a commutative diagram*

$$\begin{array}{ccccc} & & K_0\mathsf{M}(d\text{-perf}) & & \\ & {\scriptstyle \mathcal{H}'_d}\nearrow & \Big| & \searrow{\scriptstyle \mathcal{R}'} & \\ K_0\mathsf{P}_d(d\text{-perf}) & & \xrightarrow{\quad \mathcal{I}'_d \quad} & & K_0\mathsf{P}(d\text{-perf}) \\ \Big\| & & \Big|\,{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \overline{\tau}} \\ K_0\mathsf{P}_d(\mathrm{gr} \geq d) & & \xrightarrow{\quad \mathcal{I}_d \quad} & & K_0\mathsf{P}(\mathrm{gr} \geq d) \\ & {\scriptstyle \mathcal{H}_d}\searrow & \Big\downarrow & \nearrow{\scriptstyle \mathcal{R}} & \\ & & K_0\mathsf{M}(\mathrm{gr} \geq d) & & \end{array}$$

in which $\mathcal{I}_d$, $\mathcal{H}_d$, $\mathcal{R}$, $\mathcal{H}'_d$ and τ are isomorphisms, $\mathcal{I}'_d$ and $\mathcal{R}'$ are monomorphisms and $\overline{\tau}$ is an epimorphism.

Proof. Commutativity of the diagram is clear, and we have already seen in Corollary 12 that $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are isomorphisms. From this it follows that $\mathcal{I}'_d$ and $\mathcal{H}'_d$ are injective, and that $\overline{\tau}$ and τ are surjective. However, $\mathcal{H}'_d$ is clearly also surjective, since any finitely generated d-perfect module has a resolution in $\mathsf{P}_d(d\text{-perf})$, and hence $\mathcal{H}'_d$ and τ are isomorphisms. □

Note that Corollary 13 (and hence Corollary 12) actually holds when $d = 0$, but that including this case is unnecessary, as it is already stated in Corollary 9.

4. Proving the Main Theorem

Establishing the Main Theorem is a cumbersome task. We will construct an inverse to $\mathcal{I}_d \colon K_0\mathsf{P}_d(S\text{-tor}) \to K_0\mathsf{P}(S\text{-tor})$ as follows. Given a complex $Y \in \mathsf{P}(S\text{-tor})$, we choose $n \in \mathbb{Z}$ so that the shifted complex $\Sigma^n Y$ is in $\mathsf{P}_e(S\text{-tor})$ for some $e > d$. To this complex we associate an element $w_e(\Sigma^n Y) \in K_0\mathsf{P}_{e-1}(S\text{-tor})$; this is the crucial step, in which we "make a complex smaller", starting with the complex $\Sigma^n Y$ of amplitude (at most) e and ending up with the element $w_e(\Sigma^n Y)$, which, as we shall see, is represented by the difference of two complexes of amplitude (at most) $e - 1$. Repeating this process a finite number of times, we end up with an element $w_{d+1} \cdots w_e(\Sigma^n Y)$ in $K_0\mathsf{P}_d(S\text{-tor})$. This is the image of $[Y]$ under the inverse of $\mathcal{I}_d$.

4.1. **Contractions.**

Notation. Throughout Section 4.1, d denotes a non-negative integer and $S = (S_1, \dots, S_d)$ denotes a d-tuple of multiplicative systems of R.

Definition 14. Let X be a complex. A d-tuple $\alpha = (\alpha^1, \dots, \alpha^d)$ of families $\alpha^\nu = (\alpha^\nu_\ell)_{\ell \in \mathbb{Z}}$ of homomorphisms $\alpha^\nu_\ell \colon X_\ell \to X_{\ell+1}$ is an *S-contraction of X with weight* $s = (s_1, \dots, s_d) \in S_1 \times \cdots \times S_d$ if

$$\partial^X_{\ell+1} \alpha^\nu_\ell + \alpha^\nu_{\ell-1} \partial^X_\ell = s_\nu \operatorname{id}_{X_\ell}$$

for all $\ell \in \mathbb{Z}$ and $\nu = 1, \dots, d$.

In the case that $d = 0$, the concept of S-contractions is meaningless, and the property of having an S-contraction is trivially satisfied. In any case, the existence of an S-contraction of X with weight $s = (s_1, \dots, s_d)$ is equivalent to the condition that the morphisms $s_\nu \operatorname{id}_X \colon X \to X$ for $\nu = 1, \dots, d$ are null-homotopic.

Proposition 15. *Each complex $X \in \mathsf{P}(S\text{-tor})$ has an S-contraction.*

Proof. For each ν the $S_\nu^{-1}R$-complex $S_\nu^{-1}X$ is exact, bounded and consists of finitely generated projective $S_\nu^{-1}R$-modules, so the identity morphism $\operatorname{id}_{S_\nu^{-1}X}$ on $S_\nu^{-1}X$ is null-homotopic (see, for example, [5, Theorem IV.4.1]). Thus, we can find $S_\nu^{-1}R$-homomorphisms $b^\nu_\ell \colon S_\nu^{-1}X_\ell \to S_\nu^{-1}X_{\ell+1}$ such that

$$\partial^{S_\nu^{-1}X}_{\ell+1} b^\nu_\ell + b^\nu_{\ell-1} \partial^{S_\nu^{-1}X}_\ell = \operatorname{id}_{S_\nu^{-1}X_\ell}$$

for all $\ell \in \mathbb{Z}$. Writing each b^ν_ℓ in the form β^ν_ℓ / t_ν for an R-homomorphism $\beta^\nu_\ell \colon X_\ell \to X_{\ell+1}$ and some common denominator $t_\nu \in S_\nu$, we now have in $S_\nu^{-1}X_\ell$ that, for any $x \in X_\ell$,

$$(\partial^X_{\ell+1} \beta^\nu_\ell + \beta^\nu_{\ell-1} \partial^X_\ell)(x)/t_\nu = x/1.$$

Consequently, we can find $u_{\nu,x} \in S_\nu$ depending on x so that in X_ℓ,

$$u_{\nu,x}(\partial^X_{\ell+1} \beta^\nu_\ell + \beta^\nu_{\ell-1} \partial^X_\ell)(x) = u_{\nu,x} t_\nu x.$$

Since X is bounded and consists of finitely generated modules, by multiplying a finite number of $u_{\nu,x}$'s, we can obtain an element $u_\nu \in S_\nu$, independent of x and of ℓ, such that $u_\nu(\partial^X_{\ell+1} \beta^\nu_\ell + \beta^\nu_{\ell-1} \partial^X_\ell)(x) = u_\nu t_\nu x$ for all $\ell \in \mathbb{Z}$ and all $x \in X_\ell$. Setting $\alpha^\nu_\ell = u_\nu \beta^\nu_\ell$ and $s_\nu = u_\nu t_\nu$, we see that $\alpha = (\alpha^1, \dots, \alpha^d)$, where $\alpha^\nu = (\alpha^\nu_\ell)_{\ell \in \mathbb{Z}}$, is an S-contraction of X with weight $s = (s_1, \dots, s_d)$. □

Definition 16. Let X and Y be complexes in P with S-contractions α and β, respectively, and let $\phi \colon X \to Y$ be a morphism of complexes. Then α and β are said to be *compatible with* ϕ if they have the same weight and $\phi_{\ell+1} \alpha^\nu_\ell = \beta^\nu_\ell \phi_\ell$ for all $\ell \in \mathbb{Z}$ and $\nu = 1, \dots, d$.

Theorem 17 below provides an example of a situation where an S-contraction of a complex induces an S-contraction of another complex. Although the hypotheses of the theorem are very specific, the theorem turns out to be applicable in several situations.

Theorem 17. *Let X be a complex in P_e, where $e > 1$, and suppose that α is an S-contraction of X with weight s. Let $\widetilde{X}$ be another complex in P_e, and suppose that the complex $\widetilde{X}$ is identical to X except for the modules and differentials in degrees e and $e-1$. Suppose further that $\widetilde{X}_e = 0$ and that a morphism $\phi \colon X \to \widetilde{X}$ exists such that $\phi_\ell = \operatorname{id}_{X_\ell}$ for $\ell = 0, \dots, e-2$ and such that ϕ_{e-1} is surjective. Then the S-contraction α on X induces an S-contraction $\widetilde{\alpha}$ on $\widetilde{X}$ with weight s such that α*

and $\widetilde{\alpha}$ are compatible with the morphism ϕ; for $\nu = 1, \ldots, d$, $\widetilde{\alpha}^\nu$ is defined by setting $\widetilde{\alpha}^\nu_{e-2} = \phi_{e-1}\alpha^\nu_{e-2}$ and $\widetilde{\alpha}^\nu_\ell = \alpha^\nu_\ell$ for $\ell = 0, \ldots, e-3$.

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & X_e & \underset{\alpha^\nu_{e-1}}{\overset{\partial^X_e}{\rightleftarrows}} & X_{e-1} & \underset{\alpha^\nu_{e-2}}{\overset{\partial^X_{e-1}}{\rightleftarrows}} & X_{e-2} & \underset{\alpha^\nu_{e-3}}{\overset{\partial^X_{e-2}}{\rightleftarrows}} & \cdots & \rightleftarrows & X_1 & \underset{\alpha^X_0}{\overset{\partial^X_1}{\rightleftarrows}} & X_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \phi_{e-1}} & & \downarrow{\scriptstyle \mathrm{id}_{X_{e-2}}} & & & & \downarrow{\scriptstyle \mathrm{id}_{X_1}} & & \downarrow{\scriptstyle \mathrm{id}_{X_0}} & & \\
0 & \longrightarrow & 0 & \underset{0}{\overset{0}{\rightleftarrows}} & \widetilde{X}_{e-1} & \underset{\phi_{e-1}\alpha^\nu_{e-2}}{\overset{\partial^{\widetilde{X}}_{e-1}}{\rightleftarrows}} & X_{e-2} & \underset{\alpha^\nu_{e-3}}{\overset{\partial^X_{e-2}}{\rightleftarrows}} & \cdots & \rightleftarrows & X_1 & \underset{\alpha^\nu_0}{\overset{\partial^X_1}{\rightleftarrows}} & X_0 & \longrightarrow & 0
\end{array}$$

Proof. By inspection. □

Given an S-contraction α of X with weight $s = (s_1, \ldots, s_d)$ and a d-tuple $t = (t_1, \ldots, t_d) \in S_1 \times \cdots \times S_d$, we can construct an S-contraction $t\alpha$ of X with weight $st = (s_1t_1, \ldots, s_dt_d)$ by setting $t\alpha = (t_1\alpha^1, \ldots, t_d\alpha^d)$ where $t_\nu\alpha^\nu = (t_\nu\alpha^\nu_\ell)_{\ell\in\mathbb{Z}}$. We can also shift α n degrees to the left for some $n \in \mathbb{Z}$ to form an S-contraction $\Sigma^n\alpha$ of $\Sigma^n X$ with weight s by setting $\Sigma^n\alpha = (\Sigma^n\alpha^1, \ldots, \Sigma^n\alpha^d)$, where $\Sigma^n\alpha^\nu = ((\Sigma^n\alpha^\nu)_\ell)_{\ell\in\mathbb{Z}} = ((-1)^n\alpha^\nu_{\ell-n})_{\ell\in\mathbb{Z}}$.

The following theorem shows how to construct a natural S-contraction of the mapping cone of a morphism between two complexes that both have S-contractions. Recall that the mapping cone of a morphism $\phi\colon X \to Y$ is the complex $C(\phi)$ defined by $C(\phi)_\ell = Y_\ell \oplus X_{\ell-1} = (Y \oplus \Sigma X)_\ell$ and

$$\partial^{C(\phi)}_\ell = \begin{pmatrix} \partial^Y_\ell & \phi_{\ell-1} \\ 0 & -\partial^X_{\ell-1} \end{pmatrix} : \begin{matrix} Y_\ell \\ \oplus \\ X_{\ell-1} \end{matrix} \to \begin{matrix} Y_{\ell-1} \\ \oplus \\ X_{\ell-2} \end{matrix}$$

for all $\ell \in \mathbb{Z}$. The (degreewise) inclusion $Y \hookrightarrow C\phi$ and the (degreewise) projection $C(\phi) \twoheadrightarrow \Sigma X$ are both morphisms of complexes, and together they form the canonical short exact sequence

$$0 \to Y \to C(\phi) \to \Sigma X \to 0.$$

Theorem 18. *Let $\phi\colon X \to Y$ be a morphism of complexes and let α and β be S-contractions of X and Y, respectively, with weights s and t, respectively. Define for $\nu = 1, \ldots, d$ and $\ell \in \mathbb{Z}$ the homomorphism*

$$(\beta * \alpha)^\nu_\ell = \begin{pmatrix} s_\nu\beta^\nu_\ell & \beta^\nu_\ell\phi_\ell\alpha^\nu_{\ell-1} \\ 0 & -t_\nu\alpha^\nu_{\ell-1} \end{pmatrix} : C(\phi)_\ell = \begin{matrix} Y_\ell \\ \oplus \\ X_{\ell-1} \end{matrix} \to \begin{matrix} Y_{\ell+1} \\ \oplus \\ X_\ell \end{matrix} = C(\phi)_{\ell+1}.$$

Then $(\beta * \alpha) = ((\beta * \alpha)^1, \ldots, (\beta * \alpha)^d)$, where $(\beta * \alpha)^\nu = ((\beta * \alpha)^\nu_\ell)_{\ell\in\mathbb{Z}}$, is an S-contraction of the mapping cone $C(\phi)$ of ϕ with weight $st = (s_1t_1, \ldots, s_dt_d)$, and the S-contractions $s\beta$, $(\beta * \alpha)$ and $\Sigma t\alpha$ are compatible with the morphisms in the canonical exact sequence

$$0 \to Y \to C(\phi) \to \Sigma X \to 0.$$

Proof. By inspection. □

4.2. **The idea behind the proof of the Main Theorem.**

Notation. Throughout Section 4.2, d denotes a non-negative integer and $S = (S_1, \ldots, S_d)$ denotes a d-tuple of multiplicative systems of R. Furthermore, X denotes a fixed complex in $\mathsf{P}_e(S\text{-tor})$ for some integer $e > d$, and α denotes an S-contraction of X with weight $s = (s_1, \ldots, s_d) \in S_1 \times \cdots \times S_d$.

Proving the Main Theorem involves the introduction of a complex $\Delta_e(X, s)$. More specifically, $\Delta_e(X, s)$ is the complex $\Sigma^{e-d}K(s, X_e)$: that is, the Koszul complex of the sequence $s = (s_1, \ldots, s_d)$ with coefficients in X_e and shifted $e-d$ degrees to the left. For convenience we will now present an explicit description of $\Delta_e(X, s)$.

For any $\ell \in \mathbb{Z}$, let $\Upsilon(\ell)$ denote the set of ℓ-element subsets of $\{1,\ldots,d\}$: that is, the set of subsets in the form $i = \{i_1,\ldots,i_\ell\}$ where $1 \le i_1 < \cdots < i_\ell \le d$. In particular, $\Upsilon(0) = \{\emptyset\}$, $\Upsilon(d) = \{\{1,\ldots,d\}\}$ and $\Upsilon(\ell) = \emptyset$ for all $\ell \notin \{0,\ldots,d\}$. Thus, in any case, $\Upsilon(\ell)$ contains $\binom{d}{\ell}$ elements. An object $i \in \Upsilon(\ell)$ is called a *multi-index* and its elements are always denoted by $i_1,\ldots,i_\ell$ in increasing order, so that $i = \{i_1,\ldots,i_\ell\}$, where $1 \le i_1 < \cdots < i_\ell \le d$.

Definition 19. $\Delta_e(X,s)$ denotes the complex whose ℓ'th module is given by

$$\Delta_e(X,s)_\ell = \coprod_{i\in\Upsilon(e-\ell)} \Delta_e(X,s)^i_\ell, \quad \text{where} \quad \Delta_e(X,s)^i_\ell = X_e,$$

and whose ℓ'th differential $\partial^{\Delta_e(X,s)}_\ell\colon \Delta_e(X,s)_\ell \to \Delta_e(X,s)_{\ell-1}$ is given by the fact that its (j,i)-entry $(\partial^{\Delta_e(X,s)}_\ell)_{j,i}\colon \Delta_e(X,s)^i_\ell \to \Delta_e(X,s)^j_{\ell-1}$ for $i \in \Upsilon(e-\ell)$ and $j \in \Upsilon(e-\ell+1)$ is

$$(\partial^{\Delta_e(X,s)}_\ell)_{j,i} = \begin{cases} (-1)^{u+1} s_{j_u}\,\mathrm{id}_{X_e}, & \text{if } j\backslash i = \{j_u\} \\ 0, & \text{if } j \not\supseteq i \end{cases}$$

So $\Delta_e(X,s)$ is a complex whose ℓ'th module $\Delta_e(X,s)_\ell$ consists of $\binom{d}{e-\ell}$ copies of X_e and whose ℓ'th differential as a map from the i'th copy of X_e in $\Delta_e(X,s)_\ell$ to the j'th copy of X_e in $\Delta_e(X,s)_{\ell+1}$ is non-zero only when $i \subseteq j$, in which case it is multiplication by $(-1)^{u+1}s_{j_u}$ for the unique j_u which is in j and not in i. In particular, if $d=0$ the sequence s is empty and $\Delta_e(X,s)$ is the complex concentrated in degree e with $\Delta_e(X,s)_e = X_e$.

Proposition 20. *The complex $\Delta_e(X,s)$ is in $\mathsf{P}(S\text{-tor})$ and is concentrated in degrees $e,\ldots,e-d$.*

Proof. The definition clearly implies that $\Delta_e(X,s)$ is concentrated in degrees $e,\ldots,e-d$ and consists of finitely generated projective modules. Since $\Delta_e(X,s)$ is the Koszul complex of the sequence $s_1,\ldots,s_d$, the homology modules of $\Delta_e(X,s)$ are annihilated by the ideal $(s_1,\ldots,s_d)$ (see, for example, [1, Proposition 1.6.5]); in particular, the homology modules must be S_ν-torsion for $\nu = 1,\ldots,d$. □

The complex $\Delta_e(X,s)$ comes naturally equipped with an S-contraction.

Theorem 21. *For each $\ell \in \mathbb{Z}$ and each $\nu = 1,\ldots,d$, let the homomorphism $\delta_e(X,s)^\nu_\ell\colon \Delta_e(X,s)_\ell \to \Delta_e(X,s)_{\ell+1}$ be given by the fact that its (j,i)-entry for $i \in \Upsilon(e-\ell)$ and $j \in \Upsilon(e-\ell-1)$ is*

$$(\delta_e(X,s)^\nu_\ell)_{j,i} = \begin{cases} (-1)^{w+1}\,\mathrm{id}_{X_e}, & \text{if } i\backslash j = \{i_w\} = \{\nu\}, \\ 0, & \text{if } i \not\supseteq j. \end{cases}$$

Then $\delta_e(X,s) = (\delta_e(X,s)^1,\ldots,\delta_e(X,s)^d)$, where $\delta_e(X,s)^\nu = (\delta_e(X,s)^\nu_\ell)_{\ell\in\mathbb{Z}}$, is an S-contraction of $\Delta_e(X,s)$ with weight s

Proof. This is a matter of verification. For each $\nu \in \{1,\ldots,d\}$, $\ell \in \mathbb{Z}$ and $i,i' \in \Upsilon(d-\ell)$, the (i',i)-entry of $\partial^{\Delta_e(X,s)}_{\ell+1}\delta_e(X,s)^\nu_\ell$ is

$$\begin{array}{cl} s_\nu\,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } \nu \in i, \\ (-1)^{u+w}s_{i'_u}\,\mathrm{id}_{X_e}, & \text{if } i\backslash i' = \{i_w\} = \{\nu\} \text{ and } i'\backslash i = \{i'_u\}, \text{ and} \\ 0, & \text{otherwise,} \end{array}$$

whereas the (i',i)-entry of $\delta_e(X,s)^\nu_{\ell-1}\partial^{\Delta_e(X,s)}_\ell$ is

$$\begin{array}{cl} s_\nu\,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } \nu \notin i, \\ (-1)^{u+w+1}s_{i'_u}\,\mathrm{id}_{X_e}, & \text{if } i\backslash i' = \{i_w\} = \{\nu\} \text{ and } i'\backslash i = \{i'_u\}, \text{ and} \\ 0, & \text{otherwise.} \end{array}$$

Overall, we see that the (i',i)-entry of $\partial_{\ell+1}^{\Delta_e(X,s)}\delta_e(X,s)_\ell^\nu + \delta_e(X,s)_{\ell-1}^\nu\partial_\ell^{\Delta_e(X,s)}$ is $s_\nu \operatorname{id}_{X_e}$ if $i = i'$ and 0 otherwise. This is what we wanted to show. □

Definition 22. Let $\phi_e(X,\alpha)$ denote the family $(\phi_e(X,\alpha)_\ell)_{\ell\in\mathbb{Z}}$ of homomorphisms $\phi_e(X,\alpha)_\ell \colon X_\ell \to \Delta_e(X,s)_\ell = \coprod_{i\in\Upsilon(e-\ell)} X_e$ given by the fact that their i'th entries for $i \in \Upsilon(e-\ell)$ are

$$\phi_e(X,\alpha)_\ell^i = \alpha_{e-1}^{i_{e-\ell}}\alpha_{e-2}^{i_{e-\ell-1}}\cdots\alpha_\ell^{i_1}.$$

For $\ell = e$, this means that $\phi_e(X,\alpha)_e = \operatorname{id}_{X_e}$, and for $\ell \notin \{e,\dots,e-d\}$, it means that $\phi_e(X,\alpha)_\ell = 0$.

Proposition 23. *$\phi_e(X,\alpha)\colon X \to \Delta_e(X,s)$ is a morphism of complexes.*

Proof. Let $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$ and $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$. To prove that ϕ is a morphism, we need to show that $\phi_{\ell-1}\partial_\ell^X = \partial_\ell^\Delta\phi_\ell$ for all $\ell \in \mathbb{Z}$: that is, we need to verify that the j'th entry, $\alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_{\ell-1}^{j_1}\partial_\ell^X$, of the left side equals the j'th entry of $\partial_\ell^\Delta\phi_\ell$ for each $j \in \Upsilon(e-\ell+1)$. Since the (j,i)-entry of ∂_ℓ^Δ is $(-1)^{u+1}s_{j_u}\operatorname{id}_{X_e}$ whenever i is a subset of j with $j\backslash i = \{j_u\}$, that is, whenever $i = \{j_1,\dots,j_{u-1},j_{u+1},\dots,j_{e-\ell+1}\}$ for some $u \in \{1,\dots,e-\ell+1\}$, we see that the j'th coordinate of $\partial_\ell^\Delta\phi_\ell$ must be

$$\sum_{u=1}^{e-\ell+1}(-1)^{u+1}s_{j_u}\alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_{\ell+u-1}^{j_{u+1}}\alpha_{\ell+u-2}^{j_{u-1}}\cdots\alpha_\ell^{j_1}.$$

So overall, we need to show that

$$\sum_{u=1}^{e-\ell+1}(-1)^{u+1}s_{j_u}\alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_{\ell+u-1}^{j_{u+1}}\alpha_{\ell+u-2}^{j_{u-1}}\cdots\alpha_\ell^{j_1} = \alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_{\ell-1}^{j_1}\partial_\ell^X \tag{1}$$

for all $j \in \Upsilon(e-\ell+1)$. We do this by descending induction on ℓ.

When $\ell > e$, the equation clearly holds since both sides are trivial, and in the case that $\ell = e$, (1) states that $s_{j_1}\operatorname{id}_{X_e} = \alpha_{e-1}^{j_1}\partial_e^X$, which is satisfied since α is an S-contraction of X with weight s. Suppose now that $\ell < e$ is arbitrarily chosen and that (1) holds for larger values of ℓ. We then have

$$\begin{aligned}
\alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_{\ell-1}^{j_1}\partial_\ell^X &= \alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_\ell^{j_2}(s_{j_1}\operatorname{id}_{X_\ell} - \partial_{\ell+1}^X\alpha_\ell^{j_1})\\
&= s_{j_1}\alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_\ell^{j_2}\\
&\quad - (\sum_{u=2}^{e-\ell+1}(-1)^u s_{j_u}\alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_{\ell+u-1}^{j_{u+1}}\alpha_{\ell+u-2}^{j_{u-1}}\cdots\alpha_{\ell+1}^{j_2})\alpha_\ell^{j_1}\\
&= \sum_{u=1}^{e-\ell+1}(-1)^{u+1}s_{j_u}\alpha_{e-1}^{j_{e-\ell+1}}\cdots\alpha_{\ell+u-1}^{j_{u+1}}\alpha_{\ell+u-2}^{j_{u-1}}\cdots\alpha_\ell^{j_1}.
\end{aligned}$$

Here the second equality follows from the induction hypothesis. This proves (1) by induction, so ϕ is a morphism of complexes. □

Definition 24. The mapping cone of $\phi_e(X,\alpha)$ is denoted by $C_e(X,\alpha)$.

Letting $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$ and $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$, $C_e(X,\alpha)$ is the complex

$$0 \longrightarrow X_e \xrightarrow{\begin{pmatrix}\phi_e\\ -\partial_e^X\end{pmatrix}} \begin{matrix}\Delta_e\\ \oplus\\ X_{e-1}\end{matrix} \xrightarrow{\begin{pmatrix}\partial_e^\Delta & \phi_{e-1}\\ 0 & -\partial_{e-1}^X\end{pmatrix}} \begin{matrix}\Delta_{e-1}\\ \oplus\\ X_{e-2}\end{matrix} \longrightarrow \cdots$$

$$\cdots \longrightarrow \begin{matrix}\Delta_{e-d}\\ \oplus\\ X_{e-d-1}\end{matrix} \xrightarrow{\begin{pmatrix}0 & -\partial_{e-d-1}^X\end{pmatrix}} X_{e-d-2} \xrightarrow{-\partial_{e-d-2}^X} \cdots \longrightarrow X_0 \longrightarrow 0$$

concentrated in degrees $e+1,\dots,1$.

Since X and $\Delta_e(X,s)$ are equipped with S-contractions, Theorem 18 provides an S-contraction of $C_e(X,\alpha)$.

Definition 25. The S-contraction $\delta_e(X,s)*\alpha$ of $C_e(X,\alpha)$ is denoted by $\mu_e(X,\alpha)$.

Letting $\Delta \stackrel{\text{def}}{=} \Delta_e(X,s)$, $\phi \stackrel{\text{def}}{=} \phi_e(X,\alpha)$ and $\delta \stackrel{\text{def}}{=} \delta_e(X,s)$, $\mu_e(X,\alpha)$ is given by

$$\mu_e(X,\alpha)^\nu_\ell = \begin{pmatrix} s_\nu\delta^\nu_\ell & \delta^\nu_\ell\phi_\ell\alpha^\nu_{\ell-1} \\ & \\ 0 & -s_\nu\alpha^\nu_{\ell-1}\end{pmatrix} : \begin{matrix}\Delta_\ell \\ \oplus \\ X_{\ell-1}\end{matrix} \to \begin{matrix}\Delta_{\ell+1} \\ \oplus \\ X_\ell\end{matrix}$$

for each $\ell\in\mathbb{Z}$ and $\nu\in\{1,\dots,d\}$. The weight of $\mu_e(X,\alpha)$ is $s^2=(s_1^2,\dots,s_d^2)$.

Proposition 26. *$C_e(X,\alpha)$ is an object of $\mathsf{P}_{e+1}(S\text{-tor})$ concentrated in degrees $e+1,\dots,1$.*

Proof. $C_e(X,\alpha)$ is clearly concentrated in degrees $e+1,\dots,1$ and composed of finitely generated projective modules. To see that $C_e(X,\alpha)$ is homologically S-torsion, recall that the canonical short exact sequence

$$0\to\Delta_e(X,s)\to C_e(X,\alpha)\to\Sigma X\to 0 \tag{2}$$

induces the long exact sequence

$$\cdots\to \mathrm{H}_\ell(\Delta_e(X,s))\to \mathrm{H}_\ell(C_e(X,\alpha))\to \mathrm{H}_\ell(\Sigma X)\to\cdots$$

on homology. By localizing at S_ν for $\nu=1,\dots,d$ it follows that, since $\Delta_e(X,s)$ as well as ΣX are homologically S-torsion, $C_e(X,\alpha)$ must be homologically S-torsion as well. □

Definition 27. Let ∂^D_{e-1} denote the homomorphism

$$\partial^D_{e-1} = \begin{pmatrix} -\phi_e(X,\alpha)_{e-1} \\ \\ \partial^X_{e-1}\end{pmatrix} : X_{e-1} \longrightarrow \begin{matrix}\Delta_e(X,s)_{e-1} \\ \oplus \\ X_{e-2}\end{matrix} = C_e(X,\alpha)_{e-1},$$

and let $D_e(X,\alpha)$ denote the complex

$$0\longrightarrow X_{e-1}\xrightarrow{\partial^D_{e-1}} C_e(X,\alpha)_{e-1}\xrightarrow{-\partial^{C_e(X,\alpha)}_{e-1}} C_e(X,\alpha)_{e-2}\xrightarrow{-\partial^{C_e(X,\alpha)}_{e-2}}\cdots\longrightarrow C_e(X,\alpha)_1\longrightarrow 0$$

concentrated in degrees $e-1,\dots,0$.

(One verifies easily that $D_e(X,\alpha)$ indeed is a complex. It is identical to the shifted mapping cone $\Sigma^{-1}C_e(X,\alpha)$ except in degrees $e+1$ and e.)

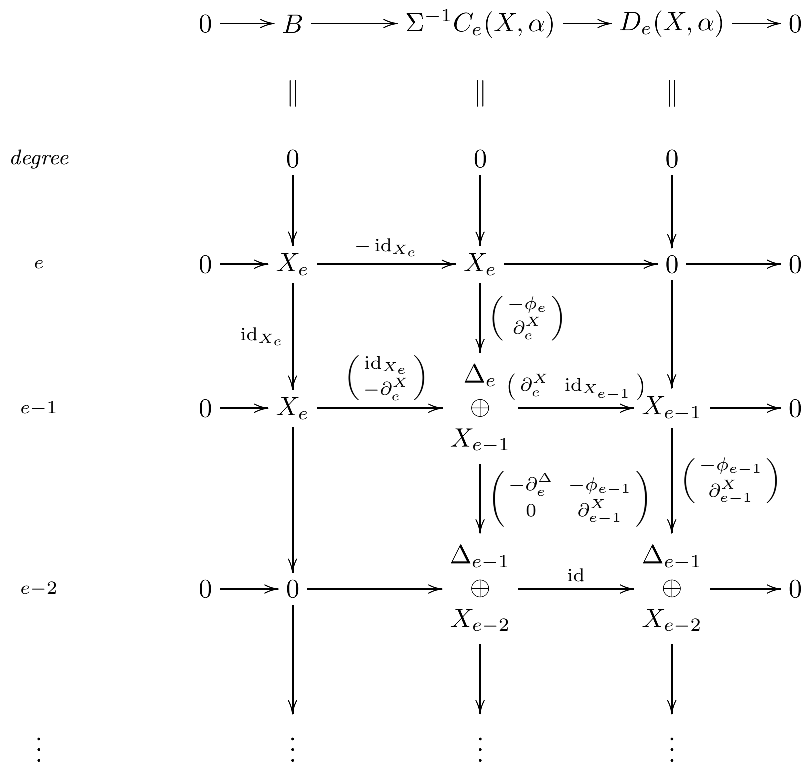
Proposition 28. *$D_e(X,\alpha)$ is an object of $\mathsf{P}_{e-1}(S\text{-tor})$.*

Proof. $D_e(X,\alpha)$ is clearly composed of finitely generated projective modules. The fact that $D_e(X,\alpha)$ is homologically S-torsion is a consequence of Theorem 29 below, from which it follows that $D_e(X,\alpha)$ is quasi-isomorphic to $\Sigma^{-1}C_e(X,\alpha)$. □

Theorem 29. *Let B denote the exact complex $0\to X_e\xrightarrow{\text{id}} X_e\to 0$ concentrated in degrees e and $e-1$. There is then an exact sequence*

$$0\to B\to\Sigma^{-1}C_e(X,\alpha)\to D_e(X,\alpha)\to 0.$$

Proof. Let $\Delta \overset{\text{def}}{=} \Delta_e(X, s)$ and $\phi \overset{\text{def}}{=} \phi_e(X, \alpha)$, and recall that $\Delta_e = X_e$ and $\phi_e = \mathrm{id}_{X_e}$. The situation is as follows.



It is straightforward to verify that the diagram commutes and that all the rows are exact. □

The morphism $\Sigma^{-1}C_e(X, \alpha) \to D_e(X, \alpha)$ from Theorem 29 is clearly in the form described in Theorem 17, so we are able to induce an S-contraction of $D_e(X, \alpha)$ with weight s^2 from the S-contraction $\Sigma^{-1}\mu_e(X, \alpha)$ on $\Sigma^{-1}C_e(X, \alpha)$.

Definition 30. The S-contraction of $D_e(X, \alpha)$ induced in the sense of Theorem 17 from $\Sigma^{-1}\mu_e(X, \alpha)$ through the morphism $\Sigma^{-1}C_e(X, \alpha) \to D_e(X, \alpha)$ from Theorem 29 is denoted by $\eta_e(X, \alpha)$.

Letting $\Delta \overset{\text{def}}{=} \Delta_e(X, s)$, $\phi \overset{\text{def}}{=} \phi_e(X, \alpha)$ and $\delta \overset{\text{def}}{=} \delta_e(X, s)$, $\eta_e(X, \alpha)$ from the above definition is given by

$$\eta_e(X, \alpha)^\nu_\ell = \begin{pmatrix} -s_\nu \delta^\nu_{\ell+1} & -\delta_{\ell+1}\phi_{\ell+1}\alpha^\nu_\ell \\ & \\ 0 & s_\nu \alpha^\nu_\ell \end{pmatrix} : \begin{matrix} \Delta_{\ell+1} \\ \oplus \\ X_\ell \end{matrix} \to \begin{matrix} \Delta_{\ell+2} \\ \oplus \\ X_{\ell+1} \end{matrix}$$

whenever $\ell = e-3, \ldots, 0$, and, as verified by a small calculation, by

$$\eta_e(X, \alpha)^\nu_{e-2} = \begin{pmatrix} -s_\nu \partial^X_e \delta^\nu_{e-1} & \alpha^\nu_{e-2}\partial^X_{e-1}\alpha^\nu_{e-2} \end{pmatrix} : \begin{matrix} \Delta_{e-1} \\ \oplus \\ X_{e-2} \end{matrix} \to X_{e-1}$$

whenever $\ell = e-2$.

From Theorem 29 and (2) in Proposition 26 it follows that

$$[X] = [\Sigma^{-1}C_e(X, \alpha)] - [\Sigma^{-1}\Delta_e(X, s)] = [D_e(X, \alpha)] - [\Sigma^{-1}\Delta_e(X, s)]$$

in $K_0\mathsf{P}_e(S\text{-tor})$. The complexes involved in the right end are both concentrated in degrees $e-1,\dots,0$. This gives us the idea of how to construct the inverse of the homomorphism $\mathcal{I}_d$ from the Main Theorem.

Definition 31. By $w_e(X,\alpha)$ we denote the element

$$w_e(X,\alpha) = [D_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)]$$

in $K_0\mathsf{P}_{e-1}(S\text{-tor})$.

The next section is devoted to showing that $w_e(X,\alpha)$ is independent of the choice of α such that we can simply write $w_e(X)$; that the map $w_e\colon \mathsf{P}_e(S\text{-tor}) \to K_0\mathsf{P}_{e-1}(S\text{-tor})$ induces a homomorphism $\mathcal{W}_e\colon K_0\mathsf{P}_e(S\text{-tor}) \to K_0\mathsf{P}_{e-1}(S\text{-tor})$; and that the $\mathcal{W}_e$'s for different e's can be combined to form an inverse of $\mathcal{I}_d$.

4.3. **Proving the Main Theorem.**

Notation. Throughout Section 4.3, d denotes a non-negative integer and $S = (S_1,\dots,S_d)$ denotes a d-tuple of multiplicative systems of R. Furthermore, X denotes a fixed complex in $\mathsf{P}_e(S\text{-tor})$ for some integer $e > d$, and α denotes an S-contraction of X with weight $s = (s_1,\dots,s_d) \in S_1 \times \cdots \times S_d$.

We begin with a collection of useful lemmas.

Lemma 32. *If*

$$0 \longrightarrow \overline{Y} \xrightarrow{\overline{\psi}} Y \xrightarrow{\psi} \widetilde{Y} \longrightarrow 0$$

is an exact sequence in $\mathsf{P}_e(S\text{-tor})$*, and if* $\overline{\beta}$*,* β *and* $\widetilde{\beta}$ *are* S*-contractions of* $\overline{Y}$*,* Y *and* $\widetilde{Y}$*, respectively, compatible with the morphisms in the above exact sequence (and thereby all having the same weight* t*), then there are exact sequences*

$$0 \to \Delta_e(\overline{Y},t) \to \Delta_e(Y,t) \to \Delta_e(\widetilde{Y},t) \to 0, \tag{3}$$

$$0 \to C_e(\overline{Y},\overline{\beta}) \to C_e(Y,\beta) \to C_e(\widetilde{Y},\widetilde{\beta}) \to 0 \quad \textit{and} \tag{4}$$

$$0 \to D_e(\overline{Y},\overline{\beta}) \to D_e(Y,\beta) \to D_e(\widetilde{Y},\widetilde{\beta}) \to 0, \tag{5}$$

proving that $w_e(Y,\beta) = w_e(\overline{Y},\overline{\beta}) + w_e(\widetilde{Y},\widetilde{\beta})$ *in* $K_0\mathsf{P}_{e-1}(S\text{-tor})$*. Furthermore, the* S*-contractions* $\delta_e(\overline{Y},t)$*,* $\delta_e(Y,t)$ *and* $\delta_e(\widetilde{Y},t)$ *are compatible with the morphisms in* (3)*; the* S*-contractions* $\mu_e(\overline{Y},\overline{\beta})$*,* $\mu_e(Y,\beta)$ *and* $\mu_e(\widetilde{Y},\widetilde{\beta})$ *are compatible with the morphisms in* (4)*; and the* S*-contractions* $\eta_e(\overline{Y},\overline{\beta})$*,* $\eta_e(Y,\beta)$ *and* $\eta_e(\widetilde{Y},\widetilde{\beta})$ *are compatible with the morphisms in* (5)*.*

Proof. According to the assumption, there is an exact sequence of modules

$$0 \longrightarrow \overline{Y}_e \xrightarrow{\overline{\psi}_e} Y_e \xrightarrow{\psi_e} \widetilde{Y}_e \longrightarrow 0,$$

which immediately induces the exact sequence in (3), because $\overline{\psi}_e$ and ψ_e clearly commute with each entry of the differentials in $\Delta_e(\overline{Y},t)$, $\Delta_e(Y,t)$ and $\Delta_e(\widetilde{Y},t)$. Since $\overline{\psi}_e$ and ψ_e also commute with each entry of the the S-contractions $\delta_e(\overline{Y},t)$, $\delta_e(Y,t)$ and $\delta_e(\widetilde{Y},t)$, these must be compatible with the morphisms in the sequence. In addition, the compatibility of the S-contractions $\overline{\beta}$, β and $\widetilde{\beta}$ with the morphisms $\overline{\psi}$ and ψ means that $\overline{\psi}_e\phi_e(\overline{Y},\overline{\beta})^i_\ell = \phi_e(Y,\beta)^i_\ell\overline{\psi}_\ell$ and $\psi_e\phi_e(Y,\beta)^i_\ell = \phi_e(\widetilde{Y},\widetilde{\beta})^i_\ell\psi_\ell$ for each $\ell \in \mathbb{Z}$ and $i \in \Upsilon(e-\ell)$, and hence that there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{Y} & \longrightarrow & Y & \longrightarrow & \widetilde{Y} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_e(\overline{Y},\overline{\beta})} & & \downarrow{\scriptstyle \phi_e(Y,\beta)} & & \downarrow{\scriptstyle \phi_e(\widetilde{Y},\widetilde{\beta})} & & \\ 0 & \longrightarrow & \Delta_e(\overline{Y},s) & \longrightarrow & \Delta_e(Y,s) & \longrightarrow & \Delta_e(\widetilde{Y},s) & \longrightarrow & 0 \end{array} \tag{6}$$

From this we induce the exact sequence of the mapping cones in (4). Straightforward calculation easily verifies that the compatibility of the S-contractions $\overline{\beta}$, β and $\widetilde{\beta}$ with the morphisms $\overline{\psi}$ and ψ, the compatibility of the S-contractions $\delta_e(\overline{Y},t)$, $\delta_e(Y,t)$ and $\delta_e(\widetilde{Y},t)$ with the morphisms in (3) and the commutativity of diagram (6) imply that the S-contractions $\mu_e(\overline{Y},\overline{\beta})$, $\mu_e(Y,\beta)$ and $\mu_e(\widetilde{Y},\widetilde{\beta})$ are compatible with the morphisms in (4).

We now claim that the exact sequence in (4) induces the exact sequence in (5). To see this, let $\overline{B}$, B and $\widetilde{B}$ denote the exact complexes $0 \to \overline{Y}_e \to \overline{Y}_e \to 0$, $0 \to Y_e \to Y_e \to 0$ and $0 \to \widetilde{Y}_e \to \widetilde{Y}_e \to 0$ from Theorem 29, concentrated in degrees e and $e-1$. These three complexes come together in a short exact sequence $0 \to \overline{B} \to B \to \widetilde{B} \to 0$, induced by the short exact sequence $0 \to \overline{Y}_e \to Y_e \to \widetilde{Y}_e \to 0$. We claim that there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{B} & \longrightarrow & B & \longrightarrow & \widetilde{B} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{-1}C_e(\overline{Y},\overline{\beta}) & \longrightarrow & \Sigma^{-1}C_e(Y,\beta) & \longrightarrow & \Sigma^{-1}C_e(\widetilde{Y},\widetilde{\beta}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_e(\overline{Y},\overline{\beta}) & \longrightarrow & D_e(Y,\beta) & \longrightarrow & D_e(\widetilde{Y},\widetilde{\beta}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The columns are exact according to Theorem 29 and the top rectangles are readily verified to be commutative. A little diagram chase now shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, making the entire diagram commutative by construction. As we have seen, the two top rows are exact, so the exactness of the bottom row follows from the 9-lemma applied in each degree. This establishes the exact sequence in (5). Once again, straightforward calculation demonstrates that the S-contractions $\eta_e(\overline{Y},\overline{\beta})$, $\eta_e(Y,\beta)$ and $\eta_e(\widetilde{Y},\widetilde{\beta})$ are compatible with the morphisms in (5).

From (3) and (5), we now obtain that

$$\begin{aligned}
w_e(Y,\beta) &= [D_e(Y,\beta)] - [\Sigma^{-1}\Delta_e(Y,t)] \\
&= [D_e(\overline{Y},\overline{\beta})] + [D_e(\widetilde{Y},\widetilde{\beta})] - [\Sigma^{-1}\Delta_e(\overline{Y},t)] - [\Sigma^{-1}\Delta_e(\widetilde{Y},t)] \\
&= w_e(\overline{Y},\overline{\beta}) + w_e(\widetilde{Y},\widetilde{\beta}),
\end{aligned}$$

and the proof is complete. □

Lemma 33. *If X is exact, then $w_e(X,\alpha) = 0$ in $K_0\mathsf{P}_{e-1}(S\text{-tor})$.*

Proof. Let $\widetilde{\partial}_{e-1}$ denote the inclusion map $\operatorname{im}\partial^X_{e-1} \hookrightarrow X_{e-2}$, and let $\widetilde{X}$ denote the complex

$$0 \longrightarrow \operatorname{im}\partial^X_{e-1} \xrightarrow{\widetilde{\partial}_{e-1}} X_{e-2} \xrightarrow{\partial^X_{e-2}} X_{e-3} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\partial^X_1} X_0 \longrightarrow 0$$

concentrated in degrees $e-1, \dots 0$. Since X is exact, $\widetilde{X}$ is exact, and it follows that $\operatorname{im}\partial^X_{e-1}$ is projective, and hence that $\widetilde{X}$ is a complex in $\mathsf{P}_{e-1}(S\text{-tor})$.

Letting B denote the exact complex $0 \to X_e \xrightarrow{\text{id}} X_e \to 0$ from Theorem 29, there is an exact sequence

$$
\begin{array}{lccccccccc}
 & 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & \widetilde{X} & \longrightarrow & 0 \\
 & & & \| & & \| & & \| & & \\
\textit{degree} & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
e & 0 & \longrightarrow & X_e & \xrightarrow{\text{id}_{X_e}} & X_e & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \text{id}_{X_e}} & & \downarrow{\scriptstyle \partial^X_e} & & \downarrow & & \\
e-1 & 0 & \longrightarrow & X_e & \xrightarrow{\partial^X_{e-1}} & X_{e-1} & \xrightarrow{\partial^X_{e-1}} & \operatorname{im}\partial^X_{e-1} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow{\scriptstyle \partial^X_{e-1}} & & \downarrow{\scriptstyle \widetilde{\partial}_{e-1}} & & \\
e-2 & 0 & \longrightarrow & 0 & \longrightarrow & X_{e-2} & \xrightarrow{\text{id}_{X_{e-2}}} & X_{e-2} & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

and we claim that there is a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{-1}\Delta_e(X,s) & \longrightarrow & \Sigma^{-1}C_e(X,\alpha) & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{-1}\Delta_e(X,s) & \longrightarrow & D_e(X,\alpha) & \longrightarrow & \widetilde{X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The columns are exact (the middle one according to Theorem 29), and the top rectangles are readily verified to be commutative. A little diagram chase shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, so that the entire diagram is commutative by construction. Now, the two top rows are exact, so the exactness of the bottom row follows from the 9-lemma applied in each degree. Thus, we have constructed an exact sequence

$$0 \to \Sigma^{-1}\Delta_e(X,s) \to D_e(X,\alpha) \to \widetilde{X} \to 0 \tag{7}$$

of complexes in $\mathsf{P}_{e-1}(S\text{-tor})$. Since $\widetilde{X}$ is exact, it follows that

$$w_e(X,\alpha) = [D_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)] = [\widetilde{X}] = 0$$

in $K_0\mathsf{P}_{e-1}(S\text{-tor})$ as desired. □

In the next lemma and the theorem that follows, we shall work with a number of similar Koszul complexes. We therefore introduce some convenient notation.

Definition 34. For $r \in S_1$, let $\Delta(r) \stackrel{\text{def}}{=} \Delta_e(X, (r, s_2, \dots, s_d))$; hence, in particular, $\Delta(s_1) = \Delta_e(X, s)$.

Lemma 35. *Suppose $r, r' \in S_1$, and define homomorphisms*

$$\pi(r,r')_\ell \colon \Delta(rr')_\ell \to \Delta(r)_\ell \quad \text{and} \quad \xi(r,r') \colon \Delta(r)_\ell \to \Delta(rr')_\ell$$

for each $\ell \in \mathbb{Z}$ by the fact that their (i', i)-entries for $i, i' \in \Upsilon(e - \ell)$ are

$$(\pi(r,r')_\ell)_{i',i} = \begin{cases} 0, & \text{if } i \neq i', \\ \mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \\ r' \,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i, \end{cases}$$

and

$$(\xi(r,r')_\ell)_{i',i} = \begin{cases} 0, & \text{if } i \neq i', \\ r' \,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \\ \mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i. \end{cases}$$

Then $\pi(r,r') = (\pi(r,r')_\ell)_{\ell \in \mathbb{Z}}$ is a morphism of complexes $\Delta(rr') \to \Delta(r)$ and $\xi(r,r') = (\xi(r,r')_\ell)_{\ell \in \mathbb{Z}}$ is a morphism of complexes $\Delta(r) \to \Delta(rr')$.

Proof. Assume that $i \in \Upsilon(e - \ell)$ and $j \in \Upsilon(e - \ell + 1)$. A direct calculation then shows that the (j, i)-entries of $\partial_\ell^{\Delta(r)} \pi(r,r')_\ell$ and $\pi(r,r')_{\ell-1} \partial_\ell^{\Delta(rr')}$ are both given by

$$\begin{array}{cl} 0, & \text{if } j \not\supseteq i, \\ (-1)^{u+1} s_{j_u} \,\mathrm{id}_{X_e}, & \text{if } j \backslash i = \{j_u\} \text{ and } 1 \in i, \\ (-1)^{u+1} s_{j_u} r' \,\mathrm{id}_{X_e}, & \text{if } j \backslash i = \{j_u\} \neq \{1\} \text{ and } 1 \notin i, \text{ and} \\ rr' \,\mathrm{id}_{X_e}, & \text{if } j \backslash i = \{j_u\} = \{1\} \text{ and } 1 \notin i. \end{array}$$

This proves that $\pi(r,r')$ is a morphism of complexes.

Similarly, a direct calculation shows that the (j, i)-entries of $\partial_\ell^{\Delta(rr')} \xi(r,r')_\ell$ and $\xi(r,r')_{\ell-1} \partial_\ell^{\Delta(r)}$ are both given by

$$\begin{array}{cl} 0, & \text{if } j \not\supseteq i, \\ (-1)^{u+1} s_{j_u} r' \,\mathrm{id}_{X_e}, & \text{if } j \backslash i = \{j_u\} \text{ and } 1 \in i, \\ (-1)^{u+1} s_{j_u} \,\mathrm{id}_{X_e}, & \text{if } j \backslash i = \{j_u\} \neq \{1\} \text{ and } 1 \notin i, \text{ and} \\ rr' \,\mathrm{id}_{X_e}, & \text{if } j \backslash i = \{j_u\} = \{1\} \text{ and } 1 \notin i. \end{array}$$

This proves that $\xi(r,r')$ is a morphism of complexes. □

We are now ready to take the first step in proving that $w_e(X, \alpha)$ is independent of the S-contraction α.

Theorem 36. *Suppose that $t = (t_1, \dots, t_d) \in S_1 \times \cdots \times S_d$ and consider the S-contraction $t\alpha = (t_1\alpha^1, \dots, t_d\alpha^d)$ of X with weight $st = (s_1t_1, \dots, s_dt_d)$. Then $w_e(X, t\alpha) = w_e(X, \alpha)$ in $K_0\mathsf{P}_{e-1}(S\text{-tor})$.*

Proof. If only we can show the equation in the case where $t_\nu = 1$ for all but one of the ν's, then the equation follows since

$$t\alpha = (t_1, \dots, t_d)\alpha = (t_1, 1 \dots, 1) \cdots (1, \dots, 1, t_d)\alpha.$$

We will therefore assume that $t = (t_1, 1, \dots, 1)$; the other cases follow similarly (since we can permute the S_ν's).

To show the desired equation, it suffices to prove that the following equations hold in $K_0\mathsf{P}_{e-1}(S\text{-tor})$.

$$[\Sigma^{-1}\Delta(s_1t_1)] = [\Sigma^{-1}\Delta(s_1)] + [\Sigma^{-1}\Delta(t_1)]. \tag{8}$$

$$[D_e(X, t\alpha)] = [D_e(X, \alpha)] + [\Sigma^{-1}\Delta(t_1)]. \tag{9}$$

Since $\Delta(1)$ is exact (being the Koszul complex of a sequence involving a unit), the first equation follows if we can show that there is an exact sequence

$$0 \longrightarrow \Delta(s_1) \xrightarrow{\left(\begin{smallmatrix}\pi(1,s_1)\\ \xi(s_1,t_1)\end{smallmatrix}\right)} \begin{matrix}\Delta(1)\\ \oplus \\ \Delta(s_1t_1)\end{matrix} \xrightarrow{(\,-\xi(1,t_1)\ \pi(t_1,s_1)\,)} \Delta(t_1) \longrightarrow 0.$$

The two matrices clearly define morphisms of complexes, since $\pi(r,r')$ and $\xi(r,r')$ are morphisms of complexes for $r,r' \in S_1$ according to Lemma 35. Exactness at $\Delta(s_1)$ and $\Delta(t_1)$ is clear since there is always one identity map involved in either of $\pi(r,r')$ and $\xi(r,r')$ for $r,r' \in S_1$. Furthermore, $\xi(1,t_1)\pi(1,s_1)$ as well as $\pi(t_1,s_1)\xi(s_1,t_1)$ are defined in degree ℓ by the fact that their (i,i')-entries for $i,i' \in \Upsilon(e-\ell)$ are

$$\begin{array}{ll} 0, & \text{if } i \neq i', \\ t_1 \,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \text{ and} \\ s_1 \,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i. \end{array}$$

To show the exactness of the sequence above, it therefore only remains to show that, for each $\ell \in \mathbb{Z}$, the kernel in degree ℓ of the second morphism is contained in the image in degree ℓ of the first. Since all (i,i')-entries of the maps involved are trivial except when $i = i'$, it suffices to consider an element (x,y) in the i-entry $\Delta(1)^i_\ell \oplus \Delta(s_1t_1)^i_\ell$ of the ℓ'th module of $\Delta(1) \oplus \Delta(s_1t_1)$. So suppose that such an element is in the kernel of the map in degree ℓ of the second morphism. If $1 \in i$, this means that $t_1x = y$, and in this case (x,y) is the image of x under the map in degree ℓ of the first morphism. If $1 \notin i$, it means that $x = s_1y$, and in this case (x,y) is the image of y under the map in degree ℓ of the first morphism. In either case, (x,y) is in the image of the map in degree ℓ of the first morphism, and hence the sequence is exact and equation (8) has been proven.

Moving on to equation (9), we first define for each $\ell \in \mathbb{Z}$ a homomorphism $\gamma_{\ell-1}\colon X_{\ell-1} \to \Delta(1)_\ell$ by letting its i'th entry for $i \in \Upsilon(e-\ell)$ be

$$\gamma^i_{\ell-1} = \begin{cases} 0, & \text{if } 1 \in i, \\ \alpha^{i_{e-\ell}}_{e-1} \cdots \alpha^{i_1}_{\ell} \alpha^1_{\ell-1}, & \text{if } 1 \notin i. \end{cases}$$

Another way of writing this is

$$\gamma_{\ell-1} = \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} \alpha^{i_{e-\ell}}_{e-1} \cdots \alpha^{i_1}_{\ell} \alpha^1_{\ell-1}.$$

We now claim that there are morphisms

$$\Phi\colon C_e(X,\alpha) \longrightarrow \begin{matrix}\Delta(1)\\ \oplus \\ C_e(X,t\alpha)\end{matrix} \quad \text{and} \quad \Psi\colon \begin{matrix}\Delta(1)\\ \oplus \\ C_e(X,t\alpha)\end{matrix} \longrightarrow \Delta(t_1)$$

given in degree ℓ by

$$\Phi_\ell = \begin{pmatrix} \pi(1,s_1)_\ell & \gamma_{\ell-1} \\ \xi(s_1,t_1)_\ell & 0 \\ 0 & \mathrm{id}_{X_{\ell-1}} \end{pmatrix} : \begin{matrix}\Delta(s_1)_\ell \\ \oplus \\ X_{\ell-1}\end{matrix} \longrightarrow \begin{matrix}\Delta(1)_\ell \\ \oplus \\ \Delta(s_1t_1)_\ell \\ \oplus \\ X_{\ell-1}\end{matrix}$$

and

$$\Psi_\ell = \begin{pmatrix} -\xi(1,t_1)_\ell & \pi(t_1,s_1)_\ell & \xi(1,t_1)_\ell\gamma_{\ell-1} \end{pmatrix} : \begin{matrix}\Delta(1)_\ell \\ \oplus \\ \Delta(s_1t_1)_\ell \\ \oplus \\ X_{\ell-1}\end{matrix} \longrightarrow \Delta(t_1)_\ell.$$

Proving that Φ and Ψ indeed are morphisms of complexes means proving that

$$\begin{pmatrix} \partial_{\ell+1}^{\Delta(1)} & 0 & 0 \\ 0 & \partial_{\ell+1}^{\Delta(s_1t_1)} & \phi_e(X,t\alpha)_\ell \\ 0 & 0 & -\partial_\ell^X \end{pmatrix} \Phi_{\ell+1} = \Phi_\ell \begin{pmatrix} \partial_{\ell+1}^{\Delta(s_1)} & \phi_e(X,\alpha)_\ell \\ 0 & -\partial_\ell^X \end{pmatrix}$$

and

$$\partial_{\ell+1}^{\Delta(t_1)} \Psi_{\ell+1} = \Psi_\ell \begin{pmatrix} \partial_{\ell+1}^{\Delta(1)} & 0 & 0 \\ 0 & \partial_{\ell+1}^{\Delta(s_1t_1)} & \phi_e(X,t\alpha)_\ell \\ 0 & 0 & -\partial_\ell^X \end{pmatrix}$$

for all $\ell \in \mathbb{Z}$. Since we already know from Lemma 35 that $\pi(r,u)$ and $\xi(r,u)$ are morphisms for $r,u \in S_1$, proving the above equations comes down to showing that the following hold for all $\ell \in \mathbb{Z}$:

$$\pi(1,s_1)_\ell \phi_e(X,\alpha)_\ell = \partial_{\ell+1}^{\Delta(1)} \gamma_\ell + \gamma_{\ell-1}\partial_\ell^X; \tag{10}$$

$$\phi_e(X,t\alpha)_\ell = \xi(s_1,t_1)_\ell \phi_e(X,\alpha)_\ell; \quad \text{and} \tag{11}$$

$$\pi(t_1,s_1)_\ell \phi_e(X,t\alpha)_\ell = \xi(1,t_1)_\ell \gamma_{\ell-1}\partial_\ell^X + \partial_{\ell+1}^{\Delta(t_1)} \xi(1,t_1)_{\ell+1}\gamma_\ell. \tag{12}$$

We verify (10) by brute force, calculating on the right hand side of the equation:

$$\begin{aligned}
\partial_{\ell+1}^{\Delta(1)} \gamma_\ell + \gamma_{\ell-1}\partial_\ell^X &= \partial_{\ell+1}^{\Delta(1)} \coprod_{\substack{j \in \Upsilon(e-\ell-1) \\ 1 \notin j}} \alpha_{e-1}^{j_{e-\ell-1}} \cdots \alpha_{\ell+1}^{j_1} \alpha_\ell^1 \\
&\quad + \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} \alpha_{e-1}^{i_{e-\ell}} \cdots \alpha_\ell^{i_1} \alpha_{\ell-1}^1 \partial_\ell^X \\
&= \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \in i}} \alpha_{e-1}^{i_{e-\ell}} \cdots \alpha_{\ell+1}^{i_2} \alpha_\ell^1 \\
&\quad + \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} \Big(\sum_{u=1}^{e-\ell} (-1)^{u+1} s_{i_u} \alpha_{e-1}^{i_{e-\ell}} \cdots \alpha_{\ell+u}^{i_{u+1}} \alpha_{\ell+u-1}^{i_{u-1}} \cdots \alpha_{\ell+1}^{i_1} \alpha_\ell^1\Big) \\
&\quad + \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} \alpha_{e-1}^{i_{e-\ell}} \cdots \alpha_\ell^{i_1} \alpha_{\ell-1}^1 \partial_\ell^X \\
&= \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \in i}} \alpha_{e-1}^{i_{e-\ell}} \cdots \alpha_\ell^{i_1} \\
&\quad + \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} \alpha_{e-1}^{i_{e-\ell}} \cdots \alpha_\ell^{i_1} (\partial_{\ell+1}^X \alpha_\ell^1 + \alpha_{\ell-1}^1 \partial_\ell^X) \\
&= \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \in i}} \alpha_{e-1}^{i_{e-\ell}} \cdots \alpha_\ell^{i_1} + \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} s_1 \alpha_{e-1}^{i_{e-\ell}} \cdots \alpha_\ell^{i_1} \\
&= \pi(1,s_1)_\ell \phi_e(X,\alpha)_\ell.
\end{aligned}$$

Here, the third equality follows from (1) in Proposition 23. This proves the equation in (10). The equation in (11) is clear, since

$$\xi(s_1,t_1)_\ell \phi_e(X,\alpha)_\ell = \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \in i}} t_1 \phi_e(X,\alpha)_\ell^i + \coprod_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} \phi_e(X,\alpha)_\ell^i = \phi_e(X,t\alpha).$$

To prove that the equation in (12) holds, we apply (10) to the right side of (12):

$$\begin{aligned}\xi(1,t_1)_\ell\gamma_{\ell-1}\partial^X_\ell + \partial^{\Delta(t_1)}_{\ell+1}\xi(1,t_1)_{\ell+1}\gamma_\ell \\ &= \xi(1,t_1)_\ell(\gamma_{\ell-1}\partial^X_\ell + \partial^{\Delta(1)}_{\ell+1}\gamma_\ell) \\ &= \xi(1,t_1)_\ell\pi(1,s_1)_\ell\phi_e(X,\alpha)_\ell.\end{aligned}$$

In contrast, applying (11) to the left side of (12) yields

$$\pi(t_1,s_1)_\ell\phi_e(X,t\alpha)_\ell = \pi(t_1,s_1)_\ell\xi(s_1,t_1)_\ell\phi_e(X,\alpha)_\ell,$$

so proving equation (12) merely requires showing that

$$\xi(1,t_1)_\ell\pi(1,s_1)_\ell = \pi(t_1,s_1)_\ell\xi(s_1,t_1)_\ell. \tag{13}$$

This, however, follows since, for $i,i' \in \Upsilon(e-\ell)$, both sides of (13) have (i,i')-entries given by

$$\begin{array}{cl} 0, & \text{if } i \neq i', \\ s_1 \operatorname{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i, \text{ and} \\ t_1 \operatorname{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i. \end{array}$$

Thus, we have verified equation (12), and we conclude that Φ and Ψ are morphisms of complexes.

We now claim that there is a short exact sequence

$$0 \longrightarrow C_e(X,\alpha) \xrightarrow{\Phi} \begin{array}{c}\Delta(1)\\ \oplus \\ C_e(X,t\alpha)\end{array} \xrightarrow{\Psi} \Delta(t_1) \longrightarrow 0. \tag{14}$$

To see that the sequence is exact at $C_e(X,\alpha)$, suppose that, for some $\ell \in \mathbb{Z}$, the element $(x,y) \in \Delta(s_1)_\ell \oplus X_{\ell-1} = C_e(X,\alpha)_\ell$ maps to 0 under Φ_ℓ: that is,

$$0 = \begin{pmatrix}\pi(1,s_1)_\ell & \gamma_{\ell-1} \\ \xi(s_1,t_1)_\ell & 0 \\ 0 & \operatorname{id}_{X_{\ell-1}}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\pi(1,s_1)_\ell(x) + \gamma_{\ell-1}(y) \\ \xi(s_1,t_1)_\ell(x) \\ y\end{pmatrix}.$$

It immediately follows that $y = 0$, and we are left with the equations $\pi(1,s_1)_\ell(x) = \xi(s_1,t_1)_\ell(x) = 0$ which imply that $x = 0$. Thus, Φ_ℓ is injective and (14) is exact at $C_e(X,\alpha)$.

To see that the sequence is exact at $\Delta(t_1)$, suppose that $x \in \Delta(t_1)^i_\ell$ for some $\ell \in \mathbb{Z}$ and $i \in \Upsilon(e-\ell)$. Then, if $1 \in i$,

$$\begin{pmatrix}-\xi(1,t_1)_\ell & \pi(t_1,s_1)_\ell & \xi(1,t_1)_\ell\gamma_{\ell-1}\end{pmatrix}\begin{pmatrix}0\\x\\0\end{pmatrix} = x,$$

and if $1 \notin i$,

$$\begin{pmatrix}-\xi(1,t_1)_\ell & \pi(t_1,s_1)_\ell & \xi(1,t_1)_\ell\gamma_{\ell-1}\end{pmatrix}\begin{pmatrix}-x\\0\\0\end{pmatrix} = x.$$

In either case, x is in the image of Ψ_ℓ, and we conclude that Ψ_ℓ is surjective and that (14) is exact at $\Delta(t_1)$.

Equation (13) clearly shows that $\Psi\Phi = 0$, so to show the exactness of (14), it only remains verify that the kernel of Ψ_ℓ is contained in the image of Φ_ℓ for all $\ell \in \mathbb{Z}$. So suppose that $(x,y,z) \in \Delta(1)_\ell \oplus \Delta(s_1t_1)_\ell \oplus X_{\ell-1} = (\Delta(1) \oplus C_e(X,t\alpha))_\ell$ maps to 0 under Ψ_ℓ: that is,

$$-\xi(1,t_1)_\ell(x) + \pi(t_1,s_1)_\ell(y) + \xi(1,t_1)_\ell\gamma_{\ell-1}(z) = 0.$$

Here $x = (x_i)_{i \in \Upsilon(e-\ell)}$ and $y = (y_i)_{i \in \Upsilon(e-\ell)}$ are $\Upsilon(e-\ell)$-tuples, so the above equation states that, for $i \in \Upsilon(e-\ell)$,

$$\begin{aligned} -t_1 x_i + y_i = 0, &\quad \text{when } 1 \in i, \text{ and} \\ -x_i + s_1 y_i + \gamma^i_{\ell-1}(z) = 0, &\quad \text{when } 1 \notin i. \end{aligned}$$

Now let $w = (w_i)_{i \in \Upsilon(e-\ell)} \in \Delta(s_1)_\ell$ be defined by $w_i = x_i$ whenever $1 \in i$ and $w_i = y_i$ whenever $1 \notin i$. Then

$$\begin{aligned} \begin{pmatrix} \pi(1,s_1)_\ell & \gamma_{\ell-1} \\ \xi(s_1,t_1)_\ell & 0 \\ 0 & \mathrm{id}_{X_{\ell-1}} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} &= \begin{pmatrix} \pi(1,s_1)_\ell(w) + \gamma_{\ell-1}(z) \\ \xi(s_1,t_1)_\ell(w) \\ z \end{pmatrix} \\ &= \begin{pmatrix} \displaystyle\sum_{\substack{i \in \Upsilon(e-\ell) \\ 1 \in i}} x_i + \sum_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} (s_1 y_i + \gamma^i_{\ell-1}(z)) \\ \displaystyle\sum_{\substack{i \in \Upsilon(e-\ell) \\ 1 \in i}} t_1 x_i + \sum_{\substack{i \in \Upsilon(e-\ell) \\ 1 \notin i}} y_i \\ z \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

This proves that (x,y,z) is in the image of Φ_ℓ. We have now proved that (14) is exact.

Denoting by B the exact complex $0 \to X_e \xrightarrow{\mathrm{id}} X_e \to 0$, we now claim that there is a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Sigma^{-1}C_e(X,\alpha) & \longrightarrow & \begin{matrix} \Sigma^{-1}\Delta(1) \\ \oplus \\ \Sigma^{-1}C_e(X,t\alpha) \end{matrix} & \longrightarrow & \Sigma^{-1}\Delta(t_1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D_e(X,\alpha) & \longrightarrow & \begin{matrix} \Sigma^{-1}\Delta(1) \\ \oplus \\ D_e(X,t\alpha) \end{matrix} & \longrightarrow & \Sigma^{-1}\Delta(t_1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The columns are exact according to Theorem 29 and the top rectangles are readily verified to be commutative. A little diagram chase shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, so that the entire diagram is commutative by construction. Now, the top row is clearly exact, and we have just seen that the middle row is exact, so the exactness of the bottom row follows from the 9-lemma applied in each degree. Thus, we have constructed an exact sequence

$$0 \longrightarrow D_e(X,\alpha) \longrightarrow \begin{matrix} \Sigma^{-1}\Delta(1) \\ \oplus \\ D_e(X,t\alpha) \end{matrix} \longrightarrow \Sigma^{-1}\Delta(t_1) \longrightarrow 0$$

in $K_0\mathsf{P}_{e-1}(S\text{-tor})$, and since $\Delta(1)$ is exact, equation (9) follows. This proves the theorem. □

We are almost ready to take the final step in proving that $w_e(X,\alpha)$ is independent of the choice of α. But first a lemma.

Lemma 37. *If Y is an exact complex in $\mathsf{P}_{e+1}(S\text{-tor})$ and β is an S-contraction of Y with weight t, then $w_e(D_{e+1}(Y,\beta),\eta_{e+1}(Y,\beta)) \in K_0\mathsf{P}_{e-1}(S\text{-tor})$ does not depend on the choice of β (but still depends on the weight t).*

Proof. Let us consider the complex $\widetilde{Y} \in \mathsf{P}_e(S\text{-tor})$, constructed from Y in the way $\widetilde{X}$ was constructed from X in Lemma 33, and equip $\widetilde{Y}$ with the S-contraction $\widetilde{\beta}$ induced from β in the sense of Theorem 17:

$$0 \longrightarrow \operatorname{im}\partial_e^Y \underset{\partial_e^Y\beta_{e-1}^\nu}{\overset{\partial_e^{\widetilde{Y}}}{\rightleftarrows}} Y_{e-1} \underset{\beta_{e-2}^\nu}{\overset{\partial_{e-1}^Y}{\rightleftarrows}} \cdots \rightleftarrows Y_1 \underset{\beta_0^\nu}{\overset{\partial_1^Y}{\rightleftarrows}} Y_0 \longrightarrow 0.$$

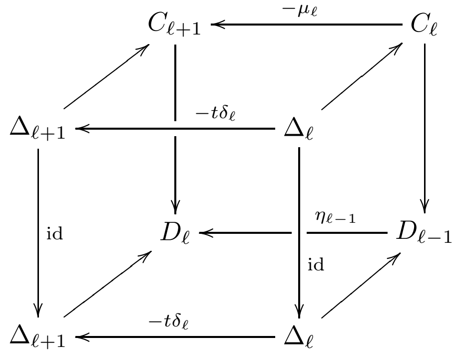
Recall that there is an S-contraction $\delta_{e+1}(Y,t)$ of $\Delta_{e+1}(Y,t)$ with weight t and an S-contraction $\mu_{e+1}(Y,\beta)$ of $C_{e+1}(Y,\beta)$ with weight t^2. According to Theorem 18, the S-contractions $\Sigma^{-1}t\delta_{e+1}(Y,t)$, $\Sigma^{-1}\mu_{e+1}(Y,\beta)$ and $t\beta$ are compatible with the morphisms in the short exact sequence

$$0 \to \Sigma^{-1}\Delta_{e+1}(Y,t) \to \Sigma^{-1}C_{e+1}(Y,\beta) \to Y \to 0. \tag{15}$$

Now, the S-contraction $\eta_{e+1}(Y,\beta)$ on $D_{e+1}(Y,\beta)$ is induced in the sense of Theorem 17 by the S-contraction $\Sigma^{-1}\mu_{e+1}(Y,\beta)$ on $\Sigma^{-1}C_{e+1}(Y,\beta)$ through the morphism $\Sigma^{-1}C_{e+1}(Y,\beta) \to D_{e+1}(Y,\beta)$. Similarly, as described above, the S-contraction $\widetilde{\beta}$ on $\widetilde{Y}$ is induced in the sense of Theorem 17 by the S-contraction β on Y through the morphism $Y \to \widetilde{Y}$. We claim that this implies that the S-contractions $\Sigma^{-1}t\delta_{e+1}(Y,t)$, $\eta_{e+1}(Y,\beta)$ and $t\widetilde{\beta}$ are compatible with the morphisms in the exact sequence

$$0 \to \Sigma^{-1}\Delta_{e+1}(Y,t) \to D_{e+1}(Y,\beta) \to \widetilde{Y} \to 0$$

from (7) in Lemma 33. This is easy: let $\Delta \overset{\text{def}}{=} \Delta_{e+1}(Y,t)$, $C \overset{\text{def}}{=} C_{e+1}(Y,\beta)$, $D \overset{\text{def}}{=} D_{e+1}(Y,\beta)$, $\delta \overset{\text{def}}{=} \delta_{e+1}(Y,t)$, $\mu \overset{\text{def}}{=} \mu_{e+1}(Y,\beta)$ and $\eta \overset{\text{def}}{=} \eta_{e+1}(Y,\beta)$. Proving, for example, that $\Sigma^{-1}t\delta$ and η are compatible with the morphism $\Sigma^{-1}\Delta \to D$ means proving the commutativity of the bottom rectangle of the following diagram for all $\ell \in \mathbb{Z}$ and $\nu = 1,\dots,d$.



The top rectangle is commutative since $\Sigma^{-1}t\delta$ and $\Sigma^{-1}\mu$ are compatible with the first morphism in (15), and the back rectangle is commutative since η is induced from $\Sigma^{-1}\mu$ in the sense of Theorem 17. We have constructed the morphism $\Sigma^{-1}\Delta \to D$ by inducing it from $\Sigma^{-1}\Delta \to \Sigma^{-1}C$ via the morphism $\Sigma^{-1}C \to D$, so the rectangles on the left and right side must also be commutative. Thus all rectangles except possibly the bottom one are commutative. Since the vertical maps are all surjective, the bottom rectangle now lifts to the top rectangle, and it follows that the bottom rectangle must be commutative. A similar argument shows that η and $t\widetilde{\beta}$ are compatible with the morphism $D \to \widetilde{Y}$.

Recalling from Lemma 33 that the exactness of Y implies the exactness of $\widetilde{Y}$, we now get, using Lemmas 32 and 33, that

$$\begin{aligned} w_e(D, \eta) &= w_e(\Sigma^{-1}\Delta, \Sigma^{-1}t\delta) + w_e(\widetilde{Y}, t\widetilde{\beta}) \\ &= w_{e+1}(\Sigma^{-1}\Delta, \Sigma^{-1}t\delta), \end{aligned}$$

which does not depend on β (but apparently still depends on t). □

Theorem 38. *The element $w_e(X, \alpha) \in K_0\mathsf{P}_{e-1}(S\text{-tor})$ does not depend on the choice of α (nor on the weight s): that is, if β is an S-contraction of X with weight t, then $w_e(X, \alpha) = w_e(X, \beta)$.*

Proof. We can assume that the weight s of α equals the weight t of β: for if this is not the case, we consider instead the S-contractions $t\alpha$ and $s\beta$ whose weights are both st, and we know from Theorem 36 that $w_e(X, \alpha) = w_e(X, t\alpha)$ and $w_e(X, s\beta) = w_e(X, \beta)$.

Consider the mapping cone $C(\mathrm{id}_X)$ of the identity morphism $\mathrm{id}_X : X \to X$ and the canonical short exact sequence

$$0 \to X \to C(\mathrm{id}_X) \to \Sigma X \to 0.$$

According to Theorem 18, the S-contractions $s\beta$, $\beta * \alpha$ and $\Sigma s\alpha$ all have weight s^2 and are compatible with the morphisms in the above sequence.

Now, the above sequence, which is a sequence in $\mathsf{P}_{e+1}(S\text{-tor})$, induces by (5) from Lemma 32 the following exact sequence in $\mathsf{P}_e(S\text{-tor})$:

$$0 \to D_{e+1}(X, s\beta) \to D_{e+1}(C(\mathrm{id}_X), \beta * \alpha) \to D_{e+1}(\Sigma X, \Sigma s\alpha) \to 0.$$

According to the same lemma, the S-contractions $\eta_{e+1}(X, s\beta)$, $\eta_{e+1}(C(\mathrm{id}_X), \beta * \alpha)$ and $\eta_{e+1}(\Sigma X, \Sigma s\alpha)$, which all have weight s^4, are compatible with the morphisms in the above sequence.

In the construction of $D_{e+1}(X, s\beta)$ we have considered X as a complex concentrated in degrees $e+1, \dots, 0$. Since X_{e+1} is the zero module, $\Delta_{e+1}(X, s^2)$ is the zero complex and $D_{e+1}(X, s\beta) = X$. Furthermore, it is straightforward to see that $\eta_{e+1}(X, s\beta)$ is the same as $s^3\beta$ considered as an S-contraction of X. It now follows from Theorem 36 and Lemma 32 that

$$\begin{aligned} w_e(X, \beta) &= w_e(X, s^3\beta) \\ &= w_e(D_{e+1}(X, s\beta), \eta_{e+1}(X, s\beta)) \\ &= w_e(D_{e+1}(C(\mathrm{id}_X), \beta * \alpha), \eta_{e+1}(C(\mathrm{id}_X), \beta * \alpha)) \\ &\quad - w_e(D_{e+1}(\Sigma X, \Sigma s\alpha), \eta_{e+1}(\Sigma X, \Sigma s\alpha)). \end{aligned}$$

Since $C(\mathrm{id}_X)$ is exact, Lemma 37 implies that the first term in the above difference does not depend on $\beta * \alpha$ and thereby not on β. The second term does not depend on β either, so it follows that the difference depends only on α. Replacing β by α, we therefore find that $w_e(X, \alpha)$ is equal to the same difference, and hence $w_e(X, \alpha) = w_e(X, \beta)$ as desired. □

Definition 39. In the light of Theorem 38, we shall write $w_e(X)$ to mean $w_e(X, \alpha)$ for any choice of S-contraction α of X.

We have now accomplished the first and hardest task in constructing an inverse to the homomorphism $\mathcal{I}_d$ from the Main Theorem. Our second task is achieved in the theorem below.

Theorem 40. *The map* $w_e\colon \mathsf{P}_e(S\text{-tor}) \to K_0\mathsf{P}_{e-1}(S\text{-tor})$ *induces a group homomorphism* $\mathcal{W}_e\colon K_0\mathsf{P}_e(S\text{-tor}) \to K_0\mathsf{P}_{e-1}(S\text{-tor})$ *defined by* $\mathcal{W}_e([X]) = w_e(X)$ *for* $X \in \mathsf{P}_e(S\text{-tor})$.

Proof. The only thing we need to show is that the relations in $K_0\mathsf{P}_e(S\text{-tor})$ are preserved under the map w_e.

If X is exact, we already know from Lemma 33 that $w_e(X) = 0$. Thus, it only remains to show that, if

$$0 \longrightarrow \overline{X} \xrightarrow{\overline{\psi}} X \xrightarrow{\psi} \widetilde{X} \to 0 \tag{16}$$

is an exact sequence in $\mathsf{P}_e(S\text{-tor})$, then $w_e(X) = w_e(\overline{X}) + w_e(\widetilde{X})$. In this case there exists a morphism $\rho\colon \Sigma^{-1}\widetilde{X} \to \overline{X}$ with the property that its mapping cone $C(\rho)$ is isomorphic to X. Now choose S-contractions $\overline{\alpha}$ and $\widetilde{\alpha}$ for $\overline{X}$ and $\widetilde{X}$, respectively, and let $\overline{s}$ and $\widetilde{s}$ denote the weights of $\overline{\alpha}$ and $\widetilde{\alpha}$, respectively. Recall that $\overline{\alpha} * \Sigma^{-1}\widetilde{\alpha}$ is an S-contraction of $C(\rho)$ with weight $\overline{s}\widetilde{s}$. We now have

$$\begin{aligned} w_e(X) &= w_e(C(\rho)) \\ &= w_e(C(\rho), \overline{\alpha} * \Sigma^{-1}\widetilde{\alpha}) \\ &= w_e(\overline{X}, \widetilde{s}\overline{\alpha}) + w_e(\widetilde{X}, \overline{s}\widetilde{\alpha}) \\ &= w_e(\overline{X}) + w_e(\widetilde{X}), \end{aligned}$$

where the third equality follows from Theorem 18 and Lemma 32. This proves the theorem. □

We are immediately able to show that our homomorphism $\mathcal{W}_e$ in fact is an isomorphism.

Theorem 41. *The group homomorphism*

$$\mathcal{I}'_{e-1}\colon K_0\mathsf{P}_{e-1}(S\text{-tor}) \to K_0\mathsf{P}_e(S\text{-tor})$$

given by $\mathcal{I}'_{e-1}([X]) = [X]$ *is an isomorphism; in fact, the inverse of* $\mathcal{I}'_{e-1}$ *is* $\mathcal{W}_e$.

Proof. If we shift the canonical exact sequence of the mapping cone $C_e(X, \alpha)$ one degree to the right, we get the exact sequence

$$0 \to \Sigma^{-1}\Delta_e(X, s) \to \Sigma^{-1}C_e(X, \alpha) \to X \to 0 \tag{17}$$

in $\mathsf{P}_e(S\text{-tor})$. Theorem 29 showed that there is an exact sequence

$$0 \to B \to \Sigma^{-1}C_e(X, \alpha) \to D_e(X, \alpha) \to 0 \tag{18}$$

in $\mathsf{P}_e(S\text{-tor})$, where B is the exact complex $0 \to X_e \xrightarrow{\text{id}} X_e \to 0$ concentrated in degrees e and $e-1$. From the exact sequences in (17) and (18) it now follows that the following holds in $K_0\mathsf{P}_e(S\text{-tor})$.

$$\begin{aligned} [X] &= [\Sigma^{-1}C_e(X, \alpha)] - [\Sigma^{-1}\Delta_e(X, s)] \\ &= [D_e(X, \alpha)] - [\Sigma^{-1}\Delta_e(X, s)] \\ &= \mathcal{I}'_{e-1}\,\mathcal{W}_e([X]). \end{aligned}$$

Conversely, suppose that Y is a complex in $\mathsf{P}_{e-1}(S\text{-tor})$ and that β is an S-contraction of Y with weight t. Then, considering Y as a complex in $\mathsf{P}_e(S\text{-tor})$, $\Delta_e(Y, t) = 0$ and $D_e(Y, \beta) = Y$, and therefore

$$[Y] = [D_e(Y, \beta)] - [\Sigma^{-1}\Delta_e(Y, t)] = \mathcal{W}_e\,\mathcal{I}'_{e-1}([Y]).$$

Thus, $\mathcal{I}'_{e-1}$ and $\mathcal{W}_e$ are inverses of each other, and the theorem is proved. □

Theorem 42 (Main Theorem)**.** *The group homomorphism*

$$\mathcal{I}_d\colon K_0\mathsf{P}_d(S\text{-tor}) \to K_0\mathsf{P}(S\text{-tor})$$

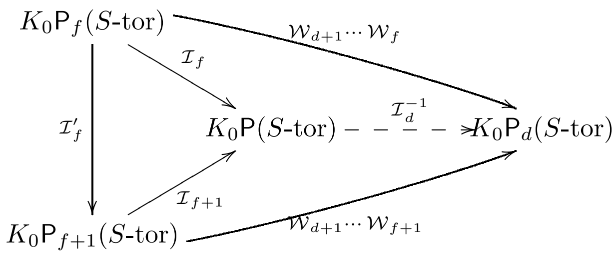
given by $\mathcal{I}_d([X]) = [X]$ *is an isomorphism.*

Proof. The sequence

$$K_0\mathsf{P}_d(S\text{-tor}) \xrightarrow{\mathcal{I}'_d} K_0\mathsf{P}_{d+1}(S\text{-tor}) \xrightarrow{\mathcal{I}'_{d+1}} \cdots$$

of Abelian groups $K_0\mathsf{P}_f(S\text{-tor})$ and homomorphisms $\mathcal{I}'_f$ for $f \geq d$ is a direct system, and it is straightforward to see that the Grothendieck group $K_0\mathsf{P}(S\text{-tor})$ together with the maps $\mathcal{I}_f\colon K_0\mathsf{P}_f(S\text{-tor}) \to K_0\mathsf{P}(S\text{-tor})$, $f \geq d$, satisfies the universal property required by a direct limit of this system (since $K_0\mathsf{P}(S\text{-tor})$ is generated by complexes concentrated in non-negative degrees). In contrast, since all the homomorphisms $\mathcal{I}'_f$ are isomorphisms according to Theorem 41, the direct limit must be isomorphic to each of the groups $K_0\mathsf{P}_f(S\text{-tor})$ and $\mathcal{I}_f$ must be an isomorphism for each $f \geq d$. □

Exploiting the property of a direct limit, we see that the inverse of $\mathcal{I}_d$ must be the homomorphism $\mathcal{I}_d^{-1}$ making the following diagram commutative.



It follows that $\mathcal{I}_d^{-1}$ is given for $Y \in \mathsf{P}(S\text{-tor})$ by

$$\mathcal{I}_d^{-1}([Y]) = (-1)^n\,\mathcal{W}_{d+1}\cdots\mathcal{W}_f([\Sigma^n Y])$$

for n and f chosen sufficiently large that $\Sigma^n Y \in \mathsf{P}_f(S\text{-tor})$.

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