# DUALITIES AND INTERSECTION MULTIPLICITIES

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Abstract. Let $R$ be a commutative, noetherian, local ring. Topological $\mathbb{Q}$–vector spaces modelled on full subcategories of the derived category of $R$ are constructed in order to study intersection multiplicities.

## 1. Introduction

Let $R$ be a commutative, noetherian, local ring and let $X$ and $Y$ be homologically bounded complexes over $R$ with finitely generated homology and supports intersecting at the maximal ideal. When the projective dimension of $X$ or $Y$ is finite, their intersection multiplicity is defined as

$$\chi(X,Y) = \chi(X \otimes^{\mathbf{L}}_R Y),$$

where $\chi(-)$ denotes the *Euler characteristic* defined as the alternating sum of the lengths of the homology modules. When $X$ and $Y$ are modules, this definition agrees with the intersection multiplicity defined by Serre [22].

The ring $R$ is said to *satisfy vanishing* when

$$\chi(X,Y) = 0 \quad \text{provided } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R.$$

If the above holds under the restriction that both complexes have finite projective dimension, $R$ is said to *satisfy weak vanishing.*

Assume, in addition, that $\dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R$ and that $R$ has prime characteristic $p$. The *Dutta multiplicity* of $X$ and $Y$ is defined when $X$ has finite projective dimension as the limit

$$\chi_\infty(X,Y) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim}(\operatorname{Supp} X)}} \chi(\mathbf{L}F^e(X), Y),$$

where $\mathbf{L}F^e$ denotes the $e$-fold composition of the left-derived Frobenius functor; the Frobenius functor $F$ was systematically used in the classical work by Peskine and Szpiro [18]. When $X$ and $Y$ are modules, $\chi_\infty(X,Y)$ is the usual Dutta multiplicity; see Dutta [6].

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and let $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ denote the full subcategory of the derived category of $R$ comprising the homologically bounded complexes with finitely generated homology and support contained in $\mathfrak{X}$. The symbols $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ denote the full subcategories of $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ comprising the complexes that are isomorphic to a complex of projective or injective modules,

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respectively. The *Grothendieck spaces* $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ are topological $\mathbb{Q}$–vector spaces modelled on these categories. The first two of these spaces were introduced in [11] but were there modelled on ordinary non-derived categories of complexes. The construction of Grothendieck spaces is similar to that of Grothendieck groups but targeted at the study of intersection multiplicities.

The main result of [11] is a diagonalization theorem in prime characteristic $p$ for the automorphism on $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ induced by the Frobenius functor. A consequence of this theorem is that every element $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ can be decomposed as

$$\alpha = \alpha^{(0)} + \alpha^{(1)} + \cdots + \alpha^{(u)},$$

where the component of degree zero describes the Dutta multiplicity, whereas the components of higher degree describe the extent to which vanishing fails to hold for the the intersection multiplicity. This paper presents (see Theorem 6.2) a similar diagonalization theorem for a functor that is analogous to the Frobenius functor and has been studied by Herzog [13]. A consequence is that every element $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ can be decomposed as

$$\beta = \beta^{(0)} + \beta^{(1)} + \cdots + \beta^{(v)},$$

where the component of degree zero describes an analog of the Dutta multiplicity, whereas the components of higher degree describe the extent to which vanishing fails to hold for the Euler form, introduced by Mori and Smith [16]. Another consequence (see Theorem 6.12) is that $R$ satisfies weak vanishing if only the Euler characteristic of homologically bounded complexes with finite-length homology changes by a factor $p^{\dim R}$ when the analogous Frobenius functor is applied.

The duality functor $(-)^* = \mathbf{R}\mathrm{Hom}_R(-,R)$ on $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ induces an automorphism on $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ which in prime characteristic $p$ is given by (see Theorem 7.5)

$$(-1)^{\operatorname{codim}\mathfrak{X}}\alpha^* = \alpha^{(0)} - \alpha^{(1)} + \cdots + (-1)^u\alpha^{(u)}.$$

Even in arbitrary characteristic, $R$ satisfies vanishing if and only if all elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ are *self-dual* in the sense that $\alpha = (-1)^{\operatorname{codim}\mathfrak{X}}\alpha^*$; and $R$ satisfies weak vanishing if all elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ are *numerically self-dual*, meaning that $\alpha - (-1)^{\operatorname{codim}\mathfrak{X}}\alpha^*$ is in the kernel of the homomorphism $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ induced by the inclusion of the underlying categories (see Theorem 7.4). Rings for which all elements of the Grothendieck spaces $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ are numerically self-dual include Gorenstein rings of dimension less than or equal to five (see Proposition 7.11) and complete intersections (see Proposition 7.7 together with [11, Example 33]).

## Notation

Throughout, $R$ denotes a commutative, noetherian, local ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Unless otherwise stated, modules and complexes are assumed to be $R$–modules and $R$–complexes, respectively.

## 2. Derived categories and functors

In this section we review notation and results from the theory of derived categories, and we introduce a new star duality and derived versions of the Frobenius functor and its natural analog. For details on the derived category and derived functors, consult [9, 12, 23].

2.1. **Derived categories.** A complex $X$ is a sequence $(X_i)_{i\in\mathbb{Z}}$ of modules equipped with a differential $(\partial_i^X)_{i\in\mathbb{Z}}$ lowering the homological degree by one. The homology complex $\mathrm{H}(X)$ of $X$ is the complex whose modules are

$$\mathrm{H}(X)_i = \mathrm{H}_i(X) = \operatorname{Ker}\partial_i^X / \operatorname{Im}\partial_{i+1}$$

and whose differentials are trivial.

A morphism of complexes $\sigma\colon X \to Y$ is a family $(\sigma_i)_{i\in\mathbb{Z}}$ of homomorphisms commuting with the differentials in $X$ and $Y$. The morphism of complexes $\sigma$ is a *quasi-isomorphism* if the induced map on homology $\mathrm{H}_i(\sigma)\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is an isomorphism in every degree. Two morphisms of complexes $\sigma, \rho\colon X \to Y$ are *homotopic* if there exists a family $(s_i)_{i\in\mathbb{Z}}$ of maps $s_i\colon X_i \to Y_{i+1}$ such that

$$\sigma_i - \rho_i = \partial_{i+1}^Y s_i + s_{i-1}\partial_i^X .$$

Homotopy yields an equivalence relation in the group $\operatorname{Hom}_R(X,Y)$ of morphisms of complexes, and the *homotopy category* $\mathsf{K}(R)$ is obtained from the category of complexes $\mathsf{C}(R)$ by declaring

$$\operatorname{Hom}_{\mathsf{K}(R)}(X,Y) = \operatorname{Hom}_{\mathsf{C}(R)}(X,Y)/\text{ homotopy}.$$

The collection $S$ of quasi-isomorphisms in the triangulated category $\mathsf{K}(R)$ form a multiplicative system of morphisms. The *derived category* $\mathsf{D}(R)$ is obtained by (categorically) localizing $\mathsf{K}(R)$ with respect to $S$. Thus, quasi-isomorphisms become isomorphisms in $\mathsf{D}(R)$; in the sequel, they are denoted $\simeq$.

Let $n$ be an integer. The symbol $\Sigma^n X$ denotes the complex $X$ shifted (or translated or suspended) $n$ degrees to the left; that is, against the direction of the differential. The modules in $\Sigma^n X$ are given by $(\Sigma^n X)_i = X_{i-n}$, and the differentials are $\partial_i^{\Sigma^n X} = (-1)^n \partial_{i-n}^X$. The symbol $\sim$ denotes isomorphisms up to a shift in the derived category.

The full subcategory of $\mathsf{D}(R)$ consisting of complexes with bounded, finitely generated homology is denoted $\mathsf{D}^{\mathrm{f}}_{\square}(R)$. Complexes from $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ are called *finite* complexes. The symbols $\mathsf{P}^{\mathrm{f}}(R)$ and $\mathsf{I}^{\mathrm{f}}(R)$ denote the full subcategories of $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ consisting of complexes that are isomorphic in the derived category to a bounded complex of projective modules and isomorphic to a bounded complex of injective modules, respectively. Note that $\mathsf{P}^{\mathrm{f}}(R)$ coincides with the full subcategory $\mathsf{F}^{\mathrm{f}}(R)$ of $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ consisting of complexes isomorphic to a complex of flat modules.

2.2. **Support.** The *spectrum* of $R$, denoted $\operatorname{Spec} R$, is the set of prime ideals of $R$. A subset $\mathfrak{X}$ of $\operatorname{Spec} R$ is *specialization-closed* if it has the property

$$\mathfrak{p} \in \mathfrak{X} \text{ and } \mathfrak{p} \subseteq \mathfrak{q} \implies \mathfrak{q} \in \mathfrak{X}$$

for all prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. A subset that is closed in the Zariski topology is, in particular, specialization-closed.

The *support* of a complex $X$ is the set

$$\operatorname{Supp} X = \left\{ \mathfrak{p} \in \operatorname{Spec} R \;\middle|\; \mathrm{H}(X_{\mathfrak{p}}) \neq 0 \right\}.$$

A *finite* complex is a complex with bounded homology and finitely generated homology modules; the support of such a complex is a closed and hence specialization-closed subset of $\operatorname{Spec} R$.

For a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, the *dimension* of $\mathfrak{X}$, denoted $\dim \mathfrak{X}$, is the usual Krull dimension of $\mathfrak{X}$. When $\dim R$ is finite, the *co-dimension* of $\mathfrak{X}$, denoted $\operatorname{codim} \mathfrak{X}$, is the number $\dim R - \dim \mathfrak{X}$. For a finitely generated module $M$, the dimension and co-dimension of $M$, denoted $\dim M$ and $\operatorname{codim} M$, are the dimension and co-dimension of the support of $M$.

For a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, the symbols $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, and $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ denote the full subcategories of $\mathsf{D}^{\mathrm{f}}_{\square}(R)$, $\mathsf{P}^{\mathrm{f}}(R)$, and $\mathsf{I}^{\mathrm{f}}(R)$, respectively, consisting of complexes whose support is contained in $\mathfrak{X}$. In the case where $\mathfrak{X}$ equals $\{\mathfrak{m}\}$, we simply write $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$, $\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ and $\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$, respectively.

2.3. **Derived functors.** A complex $P$ is said to be semi-projective if the functor $\operatorname{Hom}_R(P,-)$ sends surjective quasi-isomorphisms to surjective quasi-isomorphisms. If a complex is bounded to the right and consists of projective modules, it is semi-projective. A semi-projective resolution of $M$ is a quasi-isomorphism $\pi\colon P \to X$ where $P$ is semi-projective.

Dually, a complex $I$ is said to be semi-injective if the functor $\operatorname{Hom}_R(-,I)$ sends injective quasi-isomorphisms to surjective quasi-isomorphisms. If a complex is bounded to the left and consists of injective modules, it is semi-injective. A semi-injective resolution of $Y$ is a quasi-isomorphism $\iota\colon Y \to I$ where $I$ is semi-injective. For existence of semi-projective and semi-injective resolutions see [2].

Let $X$ and $Y$ be complexes. The left-derived tensor product $X \otimes^{\mathbf{L}}_R Y$ in $\mathsf{D}(R)$ of $X$ and $Y$ is defined by

$$P \otimes_R Y \simeq X \otimes^{\mathbf{L}}_R Y \simeq X \otimes_R Q,$$

where $P \xrightarrow{\simeq} X$ is a semi-projective resolution of $X$ and $Q \xrightarrow{\simeq} Y$ is a semi-projective resolution of $Y$. The right-derived homomorphism complex $\mathbf{R}\operatorname{Hom}_R(X,Y)$ in $\mathsf{D}(R)$ of $X$ and $Y$ is defined by

$$\operatorname{Hom}_R(P,Y) \simeq \mathbf{R}\operatorname{Hom}_R(X,Y) \simeq \operatorname{Hom}_R(X,I),$$

where $P \xrightarrow{\simeq} X$ is a semi-projective resolution of $X$ and $Y \xrightarrow{\simeq} I$ is a semi-injective resolution of $Y$. When $M$ and $N$ are modules,

$$\mathrm{H}_n(M \otimes^{\mathbf{L}}_R N) \cong \operatorname{Tor}^R_n(M,N) \quad \text{and} \quad \mathrm{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(M,N)) \cong \operatorname{Ext}^n_R(M,N)$$

for all integers $n$.

2.4. **Stability.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of $\operatorname{Spec} R$ and let $X$ be a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ and $Y$ be a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y})$. Then

$$\begin{aligned}
X \otimes^{\mathbf{L}}_R Y &\in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ or } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),\\
X \otimes^{\mathbf{L}}_R Y &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),\\
X \otimes^{\mathbf{L}}_R Y &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}),\\
X \otimes^{\mathbf{L}}_R Y &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),\\
\mathbf{R}\operatorname{Hom}_R(X,Y) &\in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ or } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}),\\
\mathbf{R}\operatorname{Hom}_R(X,Y) &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),\\
\mathbf{R}\operatorname{Hom}_R(X,Y) &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}) \text{ and}\\
\mathbf{R}\operatorname{Hom}_R(X,Y) &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}).
\end{aligned} \tag{2.4.1}$$

2.5. **Functorial isomorphisms.** Throughout, we will make use of the functorial isomorphisms stated below. As we will not need them in the most general setting, the reader should bear in mind that not all the boundedness conditions imposed on the complexes are strictly necessary. For details the reader is referred e.g., to [5, A.4] and the references therein.

Let $S$ be another commutative, noetherian, local ring. Let $K, L, M \in \mathsf{D}(R)$, let $P \in \mathsf{D}(S)$ and let $N \in \mathsf{D}(R, S)$, the derived category of $R$–$S$–bi-modules. There are the next functorial isomorphisms in $\mathsf{D}(R, S)$.

(Comm) $$M \otimes^{\mathbf{L}}_R N \xrightarrow{\simeq} N \otimes^{\mathbf{L}}_R M.$$

(Assoc) $$(M \otimes^{\mathbf{L}}_R N) \otimes^{\mathbf{L}}_S P \xrightarrow{\simeq} M \otimes^{\mathbf{L}}_R (N \otimes^{\mathbf{L}}_S P).$$

(Adjoint) $$\mathbf{R}\mathrm{Hom}_S(M \otimes^{\mathbf{L}}_R N, P) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(N, P)).$$

(Swap) $$\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(P, N)) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_S(P, \mathbf{R}\mathrm{Hom}_R(M, N)).$$

Moreover, there are the following evaluation morphisms.

(Tensor-eval) $$\sigma_{KLP}\colon\ \mathbf{R}\mathrm{Hom}_R(K, L) \otimes^{\mathbf{L}}_S P \to \mathbf{R}\mathrm{Hom}_R(K, L \otimes^{\mathbf{L}}_S P).$$

(Hom-eval) $$\rho_{PLM}\colon\ P \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(L, M) \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(P, L), M).$$

In addition,

- the morphism $\sigma_{KLP}$ is invertible if $K$ is finite, $\mathrm{H}(L)$ is bounded, and either $P \in \mathsf{P}(S)$ or $K \in \mathsf{P}(R)$; and
- the morphism $\rho_{PLM}$ is invertible if $P$ is finite, $\mathrm{H}(L)$ is bounded, and either $P \in \mathsf{P}(R)$ or $M \in \mathsf{I}(R)$.

2.6. **Dualizing complexes.** A finite complex $D$ is a *dualizing complex* for $R$ if

$$D \in \mathsf{I}^{\mathrm{f}}(R) \quad \text{and} \quad R \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D, D).$$

Dualizing complexes are essentially unique: if $D$ and $D'$ are dualizing complexes for $R$, then $D \sim D'$. To check whether a finite complex $D$ is dualizing is equivalent to checking whether

$$k \sim \mathbf{R}\mathrm{Hom}_R(k, D).$$

A dualizing complex $D$ is said to be *normalized* when $k \simeq \mathbf{R}\mathrm{Hom}_R(k, D)$. If $R$ is a Cohen–Macaulay ring of dimension $d$ and $D$ is a normalized dualizing complex, then $\mathrm{H}(D)$ is concentrated in degree $d$, and the module $\mathrm{H}_d(D)$ is the (so-called) *canonical module*; see [3, Chapter 3]. Observe that $\operatorname{Supp} D = \operatorname{Spec} R$.

If $D$ is a normalized dualizing complex for $R$, then it is isomorphic to a complex

$$0 \to D_{\dim R} \to D_{\dim R-1} \to \cdots \to D_1 \to D_0 \to 0$$

consisting of injective modules, where

$$D_i = \bigoplus_{\dim R/\mathfrak{p} = i} E_R(R/\mathfrak{p})$$

and $E_R(R/\mathfrak{p})$ is the injective hull (or envelope) of $R/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$; in particular, it follows that $D_0 = E_R(k)$.

When $R$ is a homomorphic image of a local Gorenstein ring $Q$, then the $R$–complex $\Sigma^n \mathbf{R}\mathrm{Hom}_Q(R, Q)$, where $n = \dim Q - \dim R$, is a normalized dualizing complex over $R$. In particular, it follows from Cohen's structure theorem for complete local rings that any complete ring admits a dualizing complex. Conversely, if a local ring admits a dualizing complex, then it must be a homomorphic image of a Gorenstein ring; this follows from Kawasaki's proof of Sharp's conjecture; see [14].

2.7. **Dagger duality.** Assume that $R$ admits a normalized dualizing complex $D$ and consider the duality morphism of functors

$$\mathrm{id}_{\mathsf{D}(R)} \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(-,D),D).$$

It follows essentially from (Hom-eval) that the contravariant functor

$$(-)^{\dagger} = \mathbf{R}\mathrm{Hom}_R(-,D)$$

provides a duality on the category $\mathsf{D}^{\mathsf{f}}_{\square}(R)$ which restricts to a duality between $\mathsf{P}^{\mathsf{f}}(R)$ and $\mathsf{I}^{\mathsf{f}}(R)$. This duality is sometimes referred to as *dagger duality.* According to (2.4.1), if $\mathfrak{X}$ is a specialization-closed subset of Spec $R$, then dagger duality gives a duality on $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ which restricts to a duality between $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ as described by the following commutative diagram.

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) & \underset{(-)^{\dagger}}{\overset{(-)^{\dagger}}{\rightleftarrows}} & \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \\
\uparrow & & \uparrow \\
\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) & \underset{(-)^{\dagger}}{\overset{(-)^{\dagger}}{\rightleftarrows}} & \mathsf{I}^{\mathsf{f}}(\mathfrak{X}).
\end{array}$$

Here the vertical arrows are full embeddings of categories. For more details on dagger duality, see [12].

2.8. **Foxby equivalence.** Assume that $R$ admits a normalized dualizing complex $D$ and consider the two contravariant adjoint functors

$$D \otimes^{\mathbf{L}}_R - \quad \text{and} \quad \mathbf{R}\mathrm{Hom}_R(D,-),$$

which come naturally equipped the unit and co-unit morphisms

$$\eta\colon \mathrm{id}_{\mathsf{D}(R)} \to \mathbf{R}\mathrm{Hom}_R(D, D\otimes^{\mathbf{L}}_R -) \quad \text{and} \quad \varepsilon\colon D\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,-) \to \mathrm{id}_{\mathsf{D}(R)}\,.$$

It follows essentially from an application of (Tensor-eval) and (Hom-eval) that the categories $\mathsf{P}(R)$ and $\mathsf{I}(R)$ are naturally equivalent via the above two functors. This equivalence is usually known as *Foxby equivalence* and was introduced in [1], to which the reader is referred for further details.

According to (2.4.1), for a specialization-closed subset $\mathfrak{X}$ of Spec $R$, Foxby equivalence restricts to an equivalence between $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ as described by the following diagram.

$$\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathsf{I}^{\mathsf{f}}(\mathfrak{X}).$$

2.9. **Star duality.** Consider the duality morphism of functors

$$\mathrm{id}_{\mathsf{D}(R)} \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(-, R), R).$$

From an application of (Hom-eval) it is readily seen that the functor

$$(-)^* = \mathbf{R}\mathrm{Hom}_R(-, R)$$

provides a duality on the category $\mathsf{P}^{\mathrm{f}}(R)$. According to (2.4.1), for a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, star duality restricts to a duality on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ as described by following diagram.

$$\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathsf{P}^{\mathrm{f}}(\mathfrak{X}).$$

When $R$ admits a dualizing complex $D$, the star functor can also be described in terms of the dagger and Foxby functors. Indeed, it is straightforward to show that the following three contravariant endofunctors on $\mathsf{P}^{\mathrm{f}}(R)$ are isomorphic.

$$(-)^*, \quad \mathbf{R}\mathrm{Hom}_R(D, -^{\dagger}), \quad \text{and} \quad (D \otimes^{\mathbf{L}}_R -)^{\dagger}.$$

It is equally straightforward to show that the following four contravariant endofunctors on $\mathsf{I}^{\mathrm{f}}(R)$ are isomorphic.

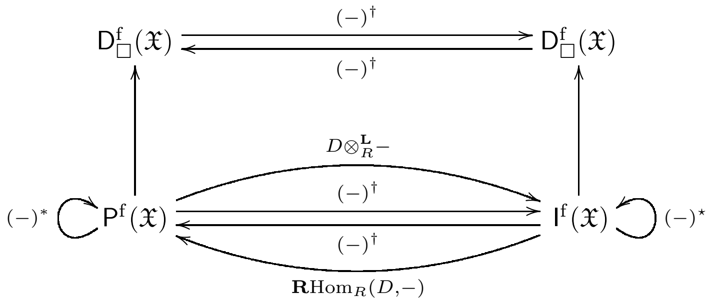
$$(-)^{\dagger\,*\,\dagger}, \quad \mathbf{R}\mathrm{Hom}_R(D, -)^{\dagger}, \quad D \otimes^{\mathbf{L}}_R (\mathbf{R}\mathrm{Hom}_R(D, -)^*) \quad \text{and} \quad D \otimes^{\mathbf{L}}_R (-)^{\dagger}.$$

They provide a duality on $\mathsf{I}^{\mathrm{f}}(R)$. In the sequel, the four isomorphic functors are denoted $(-)^{\star}$. According to (2.4.1), for a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, this new kind of star duality restricts to a duality on $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ as described by the following diagram.

$$\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \underset{(-)^{\star}}{\overset{(-)^{\star}}{\rightleftarrows}} \mathsf{I}^{\mathrm{f}}(\mathfrak{X}).$$

The dagger duality, Foxby equivalence and star duality functors fit together in the following diagram.

(2.9.1)




In the lower part of the diagram, the three types of functors, dagger, Foxby and star, always commute pairwise, and the composition of two of the three types yields a functor of the third type. For example, star duality and dagger duality always commute and compose to give Foxby equivalence, since we have

$$(-)^{*\dagger} \simeq (-)^{\dagger\star} \simeq D \otimes^{\mathbf{L}}_R - \quad \text{and} \quad (-)^{\star\dagger} \simeq (-)^{\dagger *} \simeq \mathbf{R}\mathrm{Hom}_R(D, -).$$

2.10. **Frobenius endofunctors.** Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field $k$. The endomorphism

$$f\colon R \to R \quad \text{defined by} \quad f(r) = r^p$$

for $r \in R$ is called the *Frobenius endomorphism* on $R$. The $n$-fold composition of $f$, denoted $f^n$, operates on a generic element $r \in R$ by $f^n(r) = r^{p^n}$. We let ${}^{f^n}\!R$ denote the $R$–algebra which, as a ring, is identical to $R$ but, as a module, is viewed through $f^n$. Thus, the $R$–module structure on ${}^{f^n}\!R$ is given by

$$r \cdot x = r^{p^n} x \quad \text{for } r \in R \text{ and } x \in {}^{f^n}\!R.$$

Under the present assumptions on $R$, the $R$–module ${}^{f^n}\!R$ is finitely generated (see, for example, Roberts [21, Section 7.3]).

We define two functors from the category of $R$–modules to the category of ${}^{f^n}\!R$–modules by

$$F^n(-) = - \otimes_R {}^{f^n}\!R \quad \text{and} \quad G^n(-) = \operatorname{Hom}_R({}^{f^n}\!R, -),$$

where the resulting modules are finitely generated modules with $R$–structure obtained from the ring ${}^{f^n}\!R = R$. The functor $F^n$ is called the *Frobenius functor* and has been studied by Peskine and Szpiro [18]. The functor $G^n$ has been studied by Herzog [13] and is analogous to $F^n$ in a sense that will be described below. We call this the *analogous Frobenius functor*. The $R$–structure on $F^n(M)$ is given by

$$r \cdot (m \otimes x) = m \otimes rx$$

for $r \in R$, $m \in M$ and $x \in {}^{f^n}\!R$, and the $R$–structure on $G^n(N)$ is given by

$$(r \cdot \varphi)(x) = \varphi(rx)$$

for $r \in R$, $\varphi \in \operatorname{Hom}_R({}^{f^n}\!R, N)$ and $x \in {}^{f^n}\!R$. Note that here we also have

$$(rm) \otimes x = m \otimes (r \cdot x) = m \otimes r^p x \quad \text{and} \quad r\varphi(x) = \varphi(r \cdot x) = \varphi(r^p x).$$

Peskine and Szpiro [18, Théorème (1.7)] have proven that, if $M$ has finite projective dimension, then so does $F(M)$, and Herzog [13, Satz 5.2] has proven that, if $N$ has finite injective dimension, then so does $G(N)$.

It follows by definition that the functor $F^n$ is right-exact while the functor $G^n$ is left-exact. We denote by $\mathbf{L}F^n(-)$ the left-derived of $F^n(-)$ and by $\mathbf{R}G^n(-)$ the right-derived of $G^n(-)$. When $X$ and $Y$ are $R$–complexes with semi-projective and semi-injective resolutions

$$P \xrightarrow{\simeq} X \quad \text{and} \quad Y \xrightarrow{\simeq} I,$$

respectively, these derived functors are obtained as

$$\mathbf{L}F^n(X) = P \otimes_R {}^{f^n}\!R \quad \text{and} \quad \mathbf{R}G^n(Y) = \operatorname{Hom}_R({}^{f^n}\!R, I),$$

where the resulting complexes are viewed through their ${}^{f^n}\!R$–structure, which makes them $R$–complexes since ${}^{f^n}\!R$ as a ring is just $R$. Observe that we may identify these functors with

$$\mathbf{L}F^n(X) = X \otimes^{\mathbf{L}}_R {}^{f^n}\!R \quad \text{and} \quad \mathbf{R}G^n(Y) = \mathbf{R}\operatorname{Hom}_R({}^{f^n}\!R, Y).$$

2.11. **Lemma.** *Let $R$ be a complete ring of prime characteristic and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*. Then the Frobenius functors commute with dagger and star duality in the sense that*

$$\begin{array}{lcl} \mathbf{L}F^n(-)^\dagger \simeq \mathbf{R}G^n(-^\dagger), & & \mathbf{R}G^n(-)^\dagger \simeq \mathbf{L}F^n(-^\dagger), \\ \mathbf{L}F^n(-)^* \simeq \mathbf{L}F^n(-^*) & \textit{and} & \mathbf{R}G^n(-)^\star \simeq \mathbf{R}G^n(-^\star). \end{array}$$

*Here the first row contains isomorphisms of functors between $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, while the second row contains isomorphisms of endofunctors on $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, respectively. Finally, the Frobenius functors commute with Foxby equivalence in the sense that*

$$D \otimes^{\mathbf{L}}_R \mathbf{L}F^n(-) \simeq \mathbf{R}G^n(D \otimes^{\mathbf{L}}_R -) \quad \textit{and}$$
$$\mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}G^n(-)) \simeq \mathbf{L}F^n(\mathbf{R}\mathrm{Hom}_R(D, -))$$

*as functors from $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ to $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ and from $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ to $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, respectively.*

*Proof.* Let $\varphi\colon R \to S$ be a local homomorphism making $S$ into a finitely generated $R$–module, and let $D^R$ denote a normalized dualizing complex for $R$. Then $D^S = \mathbf{R}\mathrm{Hom}_R(S, D^R)$ is a normalized dualizing complex for $S$. Pick an $R$–complex $X$ and consider the next string of natural isomorphisms.

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_S(X \otimes^{\mathbf{L}}_R S, D^S) &= \mathbf{R}\mathrm{Hom}_S(X \otimes^{\mathbf{L}}_R S, \mathbf{R}\mathrm{Hom}_R(S, D^R)) \\ &\xleftarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(X \otimes^{\mathbf{L}}_R S, D^R) \\ &\xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(S, \mathbf{R}\mathrm{Hom}_R(X, D^R)). \end{aligned}$$

Here, the two isomorphism follow from (Adjoint). The computation shows that

$$(- \otimes^{\mathbf{L}}_R S)^{\dagger_S} \simeq \mathbf{R}\mathrm{Hom}_R(S, -^{\dagger_R})$$

in $\mathsf{D}(S)$. A similar computation using the natural isomorphisms (Adjoint) and (Hom-eval) shows that

$$(-)^{\dagger_R} \otimes^{\mathbf{L}}_R S \simeq \mathbf{R}\mathrm{Hom}_R(S, -)^{\dagger_S}.$$

Under the present assumptions, the $n$-fold composition of the Frobenius endomorphism $f^n\colon R \to R$ is module-finite map. Therefore, the above isomorphisms of functors yield

$$\mathbf{L}F^n(-)^\dagger \simeq \mathbf{R}G^n(-^\dagger) \quad \text{and} \quad \mathbf{L}F^n(-^\dagger) \simeq \mathbf{R}G^n(-)^\dagger.$$

Similar considerations establish the remaining isomorphisms of functors. □

2.12. **Corollary.** *Let $R$ be a complete ring of prime characteristic and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*. Then the Frobenius functor $\mathbf{R}G^n$ is an endofunctor on $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$.*

*Proof.* From the above lemma, we learn that

$$\mathbf{R}G^n(-) \simeq (-)^\dagger \circ \mathbf{L}F^n \circ (-)^\dagger$$

and since $\mathbf{L}F^n$ is an endofunctor on $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ the conclusion is immediate. □

2.13. **Lemma.** *Let $R$ be a complete ring of prime characteristic and with perfect residue field. For complexes $X, X' \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y, Y' \in \mathsf{I}^{\mathrm{f}}(R)$ there are isomorphisms*

$$\begin{aligned}
\mathbf{L}F^n(X \otimes_R^{\mathbf{L}} X') &\simeq \mathbf{L}F^n(X) \otimes_R^{\mathbf{L}} \mathbf{L}F^n(X'),\\
\mathbf{R}G^n(X \otimes_R^{\mathbf{L}} Y) &\simeq \mathbf{L}F^n(X) \otimes_R^{\mathbf{L}} \mathbf{R}G^n(Y),\\
\mathbf{R}G^n(\mathbf{R}\mathrm{Hom}_R(X,Y)) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X), \mathbf{R}G^n(Y))\\
\mathbf{L}F^n(\mathbf{R}\mathrm{Hom}_R(X,X')) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X), \mathbf{L}F^n(X')) \quad \text{and}\\
\mathbf{L}F^n(\mathbf{R}\mathrm{Hom}_R(Y,Y')) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}G^n(Y), \mathbf{R}G^n(Y')).
\end{aligned}$$

*Proof.* We prove the first and the third isomorphism. The rest are obtained in a similar manner using Lemma 2.11 and the functorial isomorphisms.

Let $F \xrightarrow{\simeq} X$ and $F' \xrightarrow{\simeq} X'$ be finite free resolutions. Then it follows

$$\begin{aligned}
\mathbf{L}F^n(X \otimes_R^{\mathbf{L}} X') &\simeq F^n(F \otimes_R F')\\
&\simeq F^n(F) \otimes_R F^n(F')\\
&\simeq \mathbf{L}F^n(X) \otimes_R^{\mathbf{L}} \mathbf{L}F^n(X').
\end{aligned}$$

Here the first isomorphism follows as $F \otimes_R F'$ is isomorphic to $X \otimes_R^{\mathbf{L}} X'$; the second isomorphism follows from e.g., [11, Proposition 12(vi)].

From Corollary 2.12 we learn that

$$\mathbf{R}G^n(Y) \simeq (\mathbf{L}F^n(Y^\dagger))^\dagger,$$

and therefore we may compute as follows.

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X), \mathbf{R}G^n(Y)) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X), (\mathbf{L}F^n(Y^\dagger))^\dagger)\\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X) \otimes_R^{\mathbf{L}} \mathbf{L}F^n(Y^\dagger), D)\\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X \otimes_R^{\mathbf{L}} Y^\dagger), D)\\
&\simeq \mathbf{L}F^n(X \otimes_R^{\mathbf{L}} Y^\dagger)^\dagger\\
&\simeq (\mathbf{L}F^n(\mathbf{R}\mathrm{Hom}_R(X,Y)^\dagger)^\dagger\\
&\simeq \mathbf{R}G^n(\mathbf{R}\mathrm{Hom}_R(X,Y)).
\end{aligned}$$

Here the second isomorphism follows by (Adjoint); the third from the first statement in the Lemma; the fourth from definition; the fifth isomorphism follows from (Hom-eval); and the last isomorphism follows from Corollary 2.12. □

2.14. **Remark.** Any complex in $\mathsf{P}^{\mathrm{f}}(R)$ is isomorphic to a bounded complex of finitely generated, free modules, and it is well-known that the Frobenius functor acts on such a complex by simply raising the entries in the matrices representing the differentials to the $p^n$'th power. To be precise, if $X$ is a complex in the form

$$X = \cdots \longrightarrow R^m \xrightarrow{(a_{ij})} R^n \longrightarrow \cdots \longrightarrow 0,$$

then $\mathbf{L}F^n(X) = F^n(X)$ is a complex in the form

$$\mathbf{L}F^n(X) = \cdots \longrightarrow R^m \xrightarrow{(a_{ij}^{p^n})} R^n \longrightarrow \cdots \longrightarrow 0.$$

If $R$ is Cohen–Macaulay with canonical module $\omega$, then it follows from dagger duality that any complex in $\mathsf{I}^{\mathrm{f}}(R)$ is isomorphic to a complex $Y$ in the form

$$Y = 0 \longrightarrow \cdots \longrightarrow \omega^n \xrightarrow{(a_{ji})} \omega^m \longrightarrow \cdots ,$$

and $\mathbf{R}G^n$ acts on $Y$ by raising the entries in the matrices representing the differentials to the $p^n$'th power, so that $\mathbf{R}G^n(Y) = G^n(Y)$ is a complex in the form

$$\mathbf{R}G^n(Y) = 0 \longrightarrow \cdots \longrightarrow \omega^n \xrightarrow{(a_{ji}^{p^n})} \omega^m \longrightarrow \cdots .$$

## 3. Intersection multiplicities

3.1. **Serre's intersection multiplicity.** If $Z$ is a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$, then its finitely many homology modules all have finite length, and the *Euler characteristic* of $Z$ is defined by

$$\chi(Z) = \sum_i (-1)^i \operatorname{length} \mathrm{H}_i(Z).$$

Let $X$ and $Y$ be finite complexes with $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$. The *intersection multiplicity* of $X$ and $Y$ is defined by

$$\chi(X,Y) = \chi(X \otimes^{\mathbf{L}}_R Y) \quad \text{when either } X \in \mathsf{P}^{\mathrm{f}}(R) \text{ or } Y \in \mathsf{P}^{\mathrm{f}}(R).$$

In the case where $X$ and $Y$ are finitely generated modules, $\chi(X,Y)$ coincides with Serre's intersection multiplicity; see [22].

Serre's vanishing conjecture can be generalized to the statement that

(3.1.1) $$\chi(X,Y) = 0 \quad \text{if } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R$$

when either $X \in \mathsf{P}^{\mathrm{f}}(R)$ or $Y \in \mathsf{P}^{\mathrm{f}}(R)$. We will say that $R$ *satisfies vanishing* when the above holds; note that this, in general, is a stronger condition than Serre's vanishing conjecture for modules. It is known that $R$ satisfies vanishing in certain cases, for example when $R$ is regular. However, it does not hold in general, as demonstrated by Dutta, Hochster and McLaughlin [8].

If we require that both $X \in \mathsf{P}^{\mathrm{f}}(R)$ *and* $Y \in \mathsf{P}^{\mathrm{f}}(R)$, condition (3.1.1) becomes weaker. When this weaker condition is satisfies, we say that $R$ *satisfies weak vanishing.* It is known that $R$ satisfies weak vanishing in many cases, for example if $R$ is a complete intersection; see Roberts [19] or Gillet and Soulé [10]. There are, so far, no counterexamples preventing it from holding in full generality.

3.2. **Euler form.** Let $X$ and $Y$ be finite complexes with $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$. The *Euler form* of $X$ and $Y$ is defined by

$$\xi(X,Y) = \chi(\mathbf{R}\mathrm{Hom}_R(X,Y)) \quad \text{when either } X \in \mathsf{P}^{\mathrm{f}}(R) \text{ or } Y \in \mathsf{I}^{\mathrm{f}}(R).$$

In the case where $X$ and $Y$ are finitely generated modules, $\chi(X,Y)$ coincides with the Euler form introduced by Mori and Smith [16].

If $R$ admits a dualizing complex, then from Mori [17, Lemma 4.3(1) and (2)] and the definition of $(-)^\star$, we obtain

(3.2.1) $$\begin{aligned} \xi(X,Y) &= \chi(X,Y^\dagger) && \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(R) \text{ or } Y \in \mathsf{I}^{\mathrm{f}}(R), \\ \chi(X^*,Y) &= \chi(X,Y^\dagger) && \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(R), \quad \text{and} \\ \xi(X,Y^\star) &= \xi(X^\dagger,Y) && \text{whenever } Y \in \mathsf{I}^{\mathrm{f}}(R). \end{aligned}$$

Since the dagger functor does not change supports of complexes, the first formula in (3.2.1) shows that $R$ satisfies vanishing exactly when

$$\xi(X,Y) = 0 \quad \text{if } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R \tag{3.2.2}$$

when either $X \in \mathsf{P}^{\mathrm{f}}(R)$ or $Y \in \mathsf{I}^{\mathrm{f}}(R)$, and that $R$ satisfies weak vanishing exactly when (3.2.2) holds when we require both $X \in \mathsf{P}^{\mathrm{f}}(R)$ *and* $Y \in \mathsf{I}^{\mathrm{f}}(R)$.

3.3. **Dutta multiplicity.** Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $X$ and $Y$ be finite complexes with

$$\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\} \quad \text{and} \quad \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R.$$

The *Dutta multiplicity* of $X$ and $Y$ is defined by

$$\chi_\infty(X,Y) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim}(\operatorname{Supp} X)}} \chi(\mathbf{L}F^e(X), Y) \quad \text{when } X \in \mathsf{P}^{\mathrm{f}}(R).$$

When $X$ and $Y$ are finitely generated modules, $\chi_\infty(X,Y)$ coincides with the Dutta multiplicity defined in [6].

The Euler form prompts to two natural analogs of the Dutta multiplicity. We define

$$\xi_\infty(X,Y) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim}(\operatorname{Supp} Y)}} \xi(X, \mathbf{R}G^e(Y)) \quad \text{when } Y \in \mathsf{I}^{\mathrm{f}}(R), \text{ and}$$
$$\xi^\infty(X,Y) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim}(\operatorname{Supp} X)}} \xi(\mathbf{L}F^e(X), Y) \quad \text{when } X \in \mathsf{P}^{\mathrm{f}}(R).$$

We immediately note, using (3.2.1) together with Lemma 2.11, that

$$\xi_\infty(X,Y) = \chi_\infty(Y^\dagger, X) \quad \text{whenever } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}), \text{ and}$$
$$\xi^\infty(X,Y) = \chi_\infty(X^*, Y) \quad \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}).$$

## 4. Grothendieck spaces

In this section we present the definition and basic properties of Grothendieck spaces. We will introduce three types of Grothendieck spaces, two of which were introduced in [11]. The constructions in *loc. cit.* are different from the ones here but yield the same spaces.

4.1. **Complement.** For any specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, a new subset is defined by

$$\mathfrak{X}^c = \left\{ \mathfrak{p} \in \operatorname{Spec} R \,\middle|\, \mathfrak{X} \cap V(\mathfrak{p}) = \{\mathfrak{m}\} \text{ and } \dim V(\mathfrak{p}) \leqslant \operatorname{codim} \mathfrak{X} \right\}.$$

This set is engineered to be the largest subset of $\operatorname{Spec} R$ such that

$$\mathfrak{X} \cap \mathfrak{X}^c = \{\mathfrak{m}\} \quad \text{and} \quad \dim \mathfrak{X} + \dim \mathfrak{X}^c \leqslant \dim R.$$

In fact, when $\mathfrak{X}$ is closed,

$$\dim \mathfrak{X} + \dim \mathfrak{X}^c = \dim R.$$

Note that $\mathfrak{X}^c$ is specialization-closed and that $\mathfrak{X} \subseteq \mathfrak{X}^{cc}$.

4.2. **Grothendieck space.** Let $\mathfrak{X}$ be a specialization-closed subset of Spec $R$. The *Grothendieck space* of the category $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is the $\mathbb{Q}$–vector space $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ presented by elements $[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}$, one for each isomorphism class of a complex $X \in \mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, and relations

$$[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \quad \text{whenever} \quad \chi(X,-) = \chi(\widetilde{X},-)$$

as metafunctions (“functions” from a category to a set) $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \to \mathbb{Q}$.

Similarly, the Grothendieck space of the category $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ is the $\mathbb{Q}$–vector space $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ presented by elements $[Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})}$, one for each isomorphism class of a complex $Y \in \mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, and relations

$$[Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} = [\widetilde{Y}]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} \quad \text{whenever} \quad \xi(-,Y) = \xi(-,\widetilde{Y})$$

as metafunctions $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \to \mathbb{Q}$.

Finally, the Grothendieck space of the category $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ is the $\mathbb{Q}$–vector space $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ presented by elements $[Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}$, one for each isomorphism class of a complex $Z \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, and relations

$$[Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} = [\widetilde{Z}]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \quad \text{whenever} \quad \chi(-,Z) = \chi(-,\widetilde{Z})$$

as metafunctions $\mathsf{P}^{\mathsf{f}}(\mathfrak{X}^c) \to \mathbb{Q}$. Because of (3.2.1), these relations are exactly the same as the relations

$$[Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} = [\widetilde{Z}]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \quad \text{whenever} \quad \xi(Z,-) = \xi(\widetilde{Z},-)$$

as metafunctions $\mathsf{I}^{\mathsf{f}}(\mathfrak{X}^c) \to \mathbb{Q}$.

By definition of the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ there is, for each complex $Z$ in $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)$, a well-defined $\mathbb{Q}$–linear map

$$\chi(-,Z)\colon \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \mapsto \chi(X,Z).$$

We equip $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ with the *initial topology* induced by the family of maps in the above form. This topology is the coarsest topology on $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ making the above map continuous for all $Z$ in $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)$. Likewise, for each complex $Z$ in $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)$, there is a well-defined $\mathbb{Q}$–linear map

$$\xi(Z,-)\colon \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} \mapsto \xi(Z,Y),$$

and we equip $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ with the initial topology induced by the family of maps in the above form. Finally, for each complex $X$ in $\mathsf{P}^{\mathsf{f}}(\mathfrak{X}^c)$, there is a well-defined $\mathbb{Q}$–linear map

$$\chi(X,-)\colon \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad [Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \mapsto \chi(X,Z),$$

and we equip $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ with the initial topology induced by the family of maps in the above form. By (3.2.1), this topology is the same as the initial topology induced by the family of (well-defined, $\mathbb{Q}$–linear) maps in the form

$$\xi(-,Y)\colon \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad [Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \mapsto \xi(Z,Y),$$

for complexes $Y$ in $\mathsf{I}^{\mathsf{f}}(\mathfrak{X}^c)$.

It is straightforward to see that addition and scalar multiplication are continuous operations on Grothendieck spaces, making $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ topological $\mathbb{Q}$–vector spaces. We shall always consider Grothendieck spaces as topological $\mathbb{Q}$–vector spaces, so that, for example, a “homomorphism” between Grothendieck spaces means a homomorphism of topological $\mathbb{Q}$–vector spaces: that is, a continuous, $\mathbb{Q}$–linear map.

The following proposition is an improved version of [11, Proposition 2(iv) and (v)].

4.3. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$.

(*i*) *Any element in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ can be written in the form $r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ for some $r \in \mathbb{Q}$ and some $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, any element in $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ can be written in the form $s[Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}$ for some $s \in \mathbb{Q}$ and some $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, and any element in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ can be written in the form $t[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}$ for some $t \in \mathbb{Q}$ and some $Z \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. Moreover, $X$, $Y$ and $Z$ may be chosen so that*

$$\operatorname{codim}(\operatorname{Supp} X) = \operatorname{codim}(\operatorname{Supp} Y) = \operatorname{codim}(\operatorname{Supp} Z) = \operatorname{codim} \mathfrak{X}.$$

(*ii*) *For any complex $Z \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, we have the identity*

$$[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} = [\mathrm{H}(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}.$$

*In particular, the $\mathbb{Q}$–vector space $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ is generated by elements in the form $[R/\mathfrak{p}]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}$ for prime ideals $\mathfrak{p}$ in $\mathfrak{X}$.*

*Proof.* (*i*) By construction, any element $\alpha$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is a $\mathbb{Q}$–linear combination

$$\alpha = r_1[X^1]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} + \cdots + r_n[X^n]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$$

where $r_i \in \mathbb{Q}$ and $X^i \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. Since a shift of a complex changes the sign of the corresponding element in the Grothendieck space, we can assume that $r_i > 0$ for all $i$. Choosing a greatest common denominator for the $r_i$'s, we can find $r \in \mathbb{Q}$ such that

$$\alpha = r(m_1[X^1]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} + \cdots + m_n[X^n]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}) = r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})},$$

where the $m_i$'s are natural numbers and $X$ is the direct sum over $i$ of $m_i$ copies of $X^i$.

In order to prove the last statement of (*i*), choose a prime ideal $\mathfrak{p} = (a_1, \dots, a_t)$ in $\mathfrak{X}$ which is first in a chain $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t = \mathfrak{m}$ of prime ideals in $\mathfrak{X}$ of maximal length $t = \operatorname{codim} \mathfrak{X}$. Note that $\mathfrak{X} \supseteq V(\mathfrak{p})$ and that the Koszul complex $K = K(a_1, \dots, a_t)$ has support exactly equal to $V(\mathfrak{p})$. It follows that

$$\alpha = \alpha + 0 = r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} + r[K]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} - r[K]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = r[X \oplus K \oplus \Sigma K]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})},$$

where $\operatorname{codim}(\operatorname{Supp}(X \oplus K \oplus \Sigma K)) = \operatorname{codim} \mathfrak{X}$. The same argument applies to elements of $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$.

(*ii*) Any complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ is isomorphic to a bounded complex. After an appropriate shift, we may assume that $Z$ is a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ in the form

$$0 \to Z_n \to \cdots \to Z_1 \to Z_0 \to 0$$

for some natural number $n$. Since $\mathrm{H}_n(Z)$ is the kernel of the map $Z_n \to Z_{n-1}$, we can construct a short exact sequence of complexes

$$0 \to \Sigma^n \mathrm{H}_n(Z) \to Z \to Z' \to 0,$$

where $Z'$ is a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ concentrated in the same degrees as $Z$. The complex $Z'$ is exact in degree $n$, and $\mathrm{H}_i(Z') = \mathrm{H}_i(Z)$ for $i = n-1, \dots, 0$. In the Grothendieck space $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, we then have

$$[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} = [\Sigma^n \mathrm{H}_n(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} + [Z']_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}.$$

Again, $Z'$ is isomorphic to a complex concentrated in degree $n-1,\cdots,0$, so we can repeat the process a finite number of times and achieve that

$$\begin{aligned}[][Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} &= [\Sigma^n \mathrm{H}_n(Z)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} + \cdots + [\Sigma \mathrm{H}_1(Z)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} + [\mathrm{H}_0(Z)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \\ &= [\Sigma^n \mathrm{H}_n(Z) \oplus \cdots \oplus \Sigma \mathrm{H}_1(Z) \oplus \mathrm{H}_0(Z)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \\ &= [\mathrm{H}(Z)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}.\end{aligned}$$

The above analysis shows that any element of $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ can be written in the form

$$r[Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} = r\sum_i (-1)^i [\mathrm{H}_i(Z)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})},$$

which means that $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ is generated by modules. Taking a filtration of a module establishes that $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ must be generated by elements of the form $[R/\mathfrak{p}]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}$ for prime ideals $\mathfrak{p}$ in $\mathfrak{X}$. □

4.4. **Induced Euler characteristic.** The Euler characteristic $\chi\colon \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}) \to \mathbb{Q}$ induces an isomorphism[1]

$$\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}) \xrightarrow{\cong} \mathbb{Q} \quad \text{given by} \quad [Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})} \mapsto \chi(Z). \tag{4.4.1}$$

See [11] for more details. We also denote this isomorphism by $\chi$. The isomorphism means that we can identify the intersection multiplicity $\chi(X,Y)$ and the Euler form $\xi(X,Y)$ of complexes $X$ and $Y$ with elements in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$ of the form

$$[X \otimes^{\mathbf{L}}_R Y]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})} \quad \text{and} \quad [\mathbf{R}\mathrm{Hom}_R(X,Y)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})},$$

respectively.

4.5. **Induced inclusion.** Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. It is straightforward to verify that the full embeddings of $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ into $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ induce homomorphisms[2]

$$\begin{aligned}\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \quad &\text{given by} \quad [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \mapsto [X]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}, \text{ and} \\ \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \quad &\text{given by} \quad [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} \mapsto [Y]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}.\end{aligned}$$

If $\mathfrak{X}$ and $\mathfrak{X}'$ are specialization-closed subsets of $\operatorname{Spec} R$ such that that $\mathfrak{X} \subseteq \mathfrak{X}'$, then it is straightforward to verify that the full embeddings of $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ into $\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')$, $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ into $\mathsf{I}^{\mathsf{f}}(\mathfrak{X}')$ and $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ into $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}')$ induce homomorphisms

$$\begin{aligned}\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}') \quad &\text{given by} \quad [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \mapsto [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')}, \\ \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}') \quad &\text{given by} \quad [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} \mapsto [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X}')}, \text{ and} \\ \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}') \quad &\text{given by} \quad [Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \mapsto [Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}')}.\end{aligned}$$

The maps obtained in this way are called *inclusion homomorphisms*, and we shall often denote them by an overline: if $\sigma$ is an element in a Grothendieck space, then $\overline{\sigma}$ denotes the image of $\sigma$ after an application of an inclusion homomorphisms.

[1] That is, a $\mathbb{Q}$–linear homeomorphism.

[2] That is, continuous, $\mathbb{Q}$–linear maps.

4.6. **Induced tensor product and Hom.** Proposition 4.7 below shows that the left-derived tensor product functor and the right-derived Hom-functor induce bi-homomorphisms[3] on Grothendieck spaces. To clarify the contents of the proposition, let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of $\operatorname{Spec} R$ such that $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$. Proposition 4.7 states, for example, that the right-derived Hom-functor induces a bi-homomorphism

$$\mathrm{Hom}\colon \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}).$$

Given elements $\sigma \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\tau \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})$, we can, by Proposition 4.3, write

$$\sigma = r[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \quad \text{and} \quad \tau = s[Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})},$$

where $r$ and $s$ are rational numbers, $X$ is a complex in $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $Y$ is a complex in $\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})$. The bi-homomorphism above is then given by

$$(\sigma, \tau) \mapsto \mathrm{Hom}(\sigma, \tau) = rs[\mathbf{R}\mathrm{Hom}_R(X, Y)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})}. \tag{4.6.1}$$

We shall use the symbol "$\otimes$" to denote any bi-homomorphism on Grothendieck spaces induced by the left-derived tensor product and the symbol "Hom" to denote any bi-homomorphism induced by right-derived Hom-functor. Together with the isomorphism in (4.4.1) it follows that the intersection multiplicity $\chi(X, Y)$ and Euler form $\xi(X, Y)$ can be identified with elements in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$ of the form

$$\begin{array}{ccc} [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \otimes [Y]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y})}, & & [X]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \otimes [Y]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{Y})}, \\ \mathrm{Hom}([X]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}, [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})}) & \text{and} & \mathrm{Hom}([X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}, [Y]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y})}). \end{array}$$

4.7. **Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of* $\operatorname{Spec} R$ *such that* $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ *and* $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$. *The left-derived tensor product induces bi-homomorphisms as in the first column below, and the right-derived Hom-functor induces bi-homomorphisms as in the second column below.*

$$\begin{array}{ccc} \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}), & & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}), \\ \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}), & & \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}), \\ \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}), & & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}), \\ \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}), & & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}), \\ \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}) & \textit{and} & \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}). \end{array}$$

*Proof.* We verify that the map

$$\mathrm{Hom}\colon \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$$

given as in (4.6.1) is a well-defined bi-homomorphism, leaving the same verifications for the remaining maps as an easy exercise for the reader.

Therefore, assume that $X$ and $\widetilde{X}$ are complexes from $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and that $Y$ and $\widetilde{Y}$ are complexes from $\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})$ such that

$$\sigma = [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \quad \text{and} \quad \tau = [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})} = [\widetilde{Y}]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})}.$$

In order to show that the map is a well-defined $\mathbb{Q}$–bi-linear map, we are required to demonstrate that

$$[\mathbf{R}\mathrm{Hom}_R(X, Y)]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{m})} = [\mathbf{R}\mathrm{Hom}_R(\widetilde{X}, \widetilde{Y})]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{m})}.$$

[3]That is, maps that are continuous and $\mathbb{Q}$–linear in each variable.

To this end, let $Z$ be an arbitrary complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\{\mathfrak{m}\}^c) = \mathsf{D}^{\mathrm{f}}_{\square}(R)$. We want to show that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)) = \xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},\widetilde{Y})).$$

Without loss of generality, we may assume that $R$ is complete; in particular, we may assume that $R$ admits a normalized dualizing complex. Observe that

$$Z \otimes_R X \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \subseteq \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y}^c) \quad \text{and} \quad Z \otimes^{\mathbf{L}}_R Y^{\dagger} \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y}) \subseteq \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c).$$

Applying (3.2.1), (Hom-eval) and (Assoc), we learn that

$$\begin{aligned} \xi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)) &= \chi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)^{\dagger}) \\ &= \chi(Z, X \otimes^{\mathbf{L}}_R Y^{\dagger}) \\ &= \chi(X, Z \otimes^{\mathbf{L}}_R Y^{\dagger}). \end{aligned} \tag{4.7.1}$$

A similar computation shows that $\xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X}, Y)) = \chi(\widetilde{X}, Z \otimes^{\mathbf{L}}_R Y^{\dagger})$, and since $[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$, we conclude that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)) = \xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},Y)).$$

An application of (Adjoint) yields that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},Y)) = \xi(Z \otimes^{\mathbf{L}}_R \widetilde{X}, Y),$$

and similarly $\xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},\widetilde{Y})) = \xi(Z \otimes^{\mathbf{L}}_R \widetilde{X}, \widetilde{Y})$. Since $[Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})} = [\widetilde{Y}]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})}$, we conclude that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},Y)) = \xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},\widetilde{Y})).$$

Thus, we have that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)) = \xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},\widetilde{Y})),$$

which establishes well-definedness.

By definition, the induced Hom-map is $\mathbb{Q}$–linear. To establish that it is continuous in, say, the first variable it suffices for fixed $\tau \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})$ to show that, to every $\varepsilon > 0$ and every complex $Z \in \mathsf{D}^{\mathrm{f}}_{\square}(\{\mathfrak{m}\}^c) = \mathsf{D}^{\mathrm{f}}_{\square}(R)$, there exists a $\delta > 0$ and a complex $Z' \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$ such that

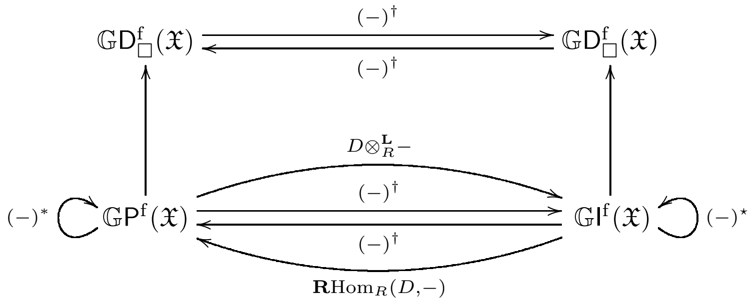
$$|\chi(\sigma, Z')| < \delta \implies |\xi(Z, \mathrm{Hom}(\sigma,\tau))| < \varepsilon.$$

We can write $\tau = r[Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})}$ for an $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y})$ and a rational number $r > 0$. According to (4.7.1), the implication above is then achieved with $Z' = Z \otimes^{\mathbf{L}}_R Y^{\dagger}$ and $\delta = \varepsilon / r$. Continuity in the second variable is shown by similar arguments. □

In Proposition 4.8 below, we will show that the dagger, Foxby and star functors from diagram (2.9.1) induce isomorphisms of Grothendieck spaces. We shall denote the isomorphisms induced by the star and dagger duality functors by the same symbol as the original functor, whereas the isomorphisms induced by the Foxby functors will be denoted according to Proposition 4.7 by $D \otimes -$ and $\mathrm{Hom}(D, -)$. In this way, for example,

$$[X]^{\dagger}_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = [X^{\dagger}]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}, \quad [X]^{*}_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = [X^{*}]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \quad \text{and} \quad D \otimes [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = [D \otimes^{\mathbf{L}}_R X]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}.$$

4.8. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, and assume that $R$ admits a dualizing complex. The functors from diagram* (2.9.1) *induce isomorphisms of Grothendieck spaces as described by the horizontal and circular arrows in the following commutative diagram.*




*Proof.* The fact that the dagger, star and Foxby functors induce homomorphisms on Grothendieck spaces follows immediately from Proposition 4.7. The fact that the induced homomorphisms are isomorphisms follows immediately from 2.7, 2.8 and 2.9, since the underlying functors define dualities or equivalences of categories. □

4.9. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$ *and consider the following elements of Grothendieck spaces.*

$$\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}), \quad \beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}), \quad \gamma \in \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^{c}) \quad \textit{and} \quad \sigma \in \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}).$$

*Then $\sigma^{\dagger} = \sigma$ holds in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$, and so do the following identities.*

$$\alpha \otimes \gamma = \mathrm{Hom}(\gamma, \alpha^{\dagger}) = \mathrm{Hom}(\alpha, \gamma^{\dagger}) = \mathrm{Hom}(\alpha^{*}, \gamma)$$
$$\mathrm{Hom}(\alpha, \gamma) = \alpha \otimes \gamma^{\dagger} = \mathrm{Hom}(\gamma, D \otimes \alpha) = \alpha^{*} \otimes \gamma$$
$$\mathrm{Hom}(\gamma, \beta) = \beta^{\dagger} \otimes \gamma = \mathrm{Hom}(\mathrm{Hom}(D, \beta), \gamma)$$
$$\mathrm{Hom}(\beta^{\dagger}, \gamma) = \mathrm{Hom}(\gamma^{\dagger}, \beta) = \mathrm{Hom}(D, \beta) \otimes \gamma = \mathrm{Hom}(\gamma, \beta^{\star})$$

*Proof.* Recall from 2.9 that the Foxby functors can be written as the composition of a star and a dagger functor. All identities follow from the formulas in (3.2.1). The formula for $\sigma$ is a consequence of the first formula in (3.2.1) in the case $X = R$. □

4.10. **Frobenius endomorphism.** Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$, and let $n$ be a non-negative integer. The derived Frobenius endofunctor $\mathbf{L}F^{n}$ on $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ induces an endomorphism[4] on $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, which will be denoted $F^{n}_{\mathfrak{X}}$; see [11] for further details. It is given for a complex $X \in \mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ by

$$F^{n}_{\mathfrak{X}}([X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}) = [\mathbf{L}F^{n}(X)]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}.$$

Let

$$\Phi^{n}_{\mathfrak{X}} = \frac{1}{p^{n \operatorname{codim} \mathfrak{X}}} F^{n}_{\mathfrak{X}} \colon \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}).$$

According to [11, Theorem 19], the endomorphism $\Phi^{n}_{\mathfrak{X}}$ is diagonalizable.

[4]That is, a continuous, $\mathbb{Q}$–linear operator.

In Lemma 2.11, we established that the functor $\mathbf{R}G^n$ is an endofunctor on $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ which can be written as

$$\mathbf{R}G^n(-) = (-)^\dagger \circ \mathbf{L}F^n \circ (-)^\dagger.$$

Thus, $\mathbf{R}G^n$ is composed of functors that induce homomorphisms on Grothendieck spaces, and hence it too induces a homomorphism $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. We denote this endomorphism on $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ by $G^n_{\mathfrak{X}}$. It is given for a complex $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ by

$$G^n_{\mathfrak{X}}([Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}) = [\mathbf{R}G^n(Y)]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}.$$

Let

$$\Psi^n_{\mathfrak{X}} = \frac{1}{p^{n \operatorname{codim} \mathfrak{X}}} G^n_{\mathfrak{X}} \colon \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}).$$

Theorem 6.2 shows that $\Psi^n_{\mathfrak{X}}$ also is a diagonalizable automorphism.

For complexes $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ we shall write $\Phi^n_{\mathfrak{X}}(X)$ and $\Psi^n_{\mathfrak{X}}(Y)$ instead of $\Phi^n_{\mathfrak{X}}([X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})})$ and $\Psi^n_{\mathfrak{X}}([Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})})$, respectively. The isomorphism in (4.4.1) together with Proposition 4.7 shows that the Dutta multiplicity $\chi_\infty(X, Y)$ and its two analogs $\xi_\infty(X, Y)$ and $\xi^\infty(X, Y)$ from Section 3.3 can be identified with elements in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{m})$ of the form

$$\lim_{e\to\infty} (\Phi^e_{\mathfrak{X}}(X) \otimes [Y]_{\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{Y})}), \quad \lim_{e\to\infty} \operatorname{Hom}([X]_{\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{X})}, \Psi^e_{\mathfrak{Y}}(Y)) \text{ and } \lim_{e\to\infty} \operatorname{Hom}(\Phi^e_{\mathfrak{X}}(X), [Y]_{\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{Y})}).$$

## 5. Vanishing

5.1. **Vanishing.** Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and consider an element $\alpha$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, an element $\beta$ in $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and an element $\gamma$ in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{X})$. The *dimensions* of $\alpha$, $\beta$ and $\gamma$ are defined as

$$\dim \alpha = \inf \left\{ \dim(\operatorname{Supp} X) \,\middle|\, \alpha = r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \text{ for some } r \in \mathbb{Q} \text{ and } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \right\},$$
$$\dim \beta = \inf \left\{ \dim(\operatorname{Supp} Y) \,\middle|\, \alpha = s[Y]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \text{ for some } s \in \mathbb{Q} \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \right\} \text{ and}$$
$$\dim \gamma = \inf \left\{ \dim(\operatorname{Supp} Z) \,\middle|\, \gamma = t[Z]_{\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{X})} \text{ for some } t \in \mathbb{Q} \text{ and } Z \in \mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{X}) \right\}.$$

In particular, the dimension of an element in a Grothendieck space is $-\infty$ if and only if the element is trivial. We say that $\alpha$ *satisfies vanishing* if

$$\alpha \otimes \sigma = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{m}) \text{ for all } \sigma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{X}^c) \text{ with } \dim \sigma < \operatorname{codim} \mathfrak{X},$$

and that $\alpha$ *satisfies weak vanishing* if

$$\overline{\alpha \otimes \tau} = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{m}) \text{ for all } \tau \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \text{ with } \dim \tau < \operatorname{codim} \mathfrak{X}.$$

Similarly, we say that $\beta$ *satisfies vanishing* if

$$\operatorname{Hom}(\sigma, \beta) = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{m}) \text{ for all } \sigma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{X}^c) \text{ with } \dim \sigma < \operatorname{codim} \mathfrak{X},$$

and that $\beta$ *satisfies weak vanishing* if

$$\overline{\operatorname{Hom}(\tau, \beta)} = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\Box}(\mathfrak{m}) \text{ for all } \tau \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \text{ with } \dim \tau < \operatorname{codim} \mathfrak{X}.$$

The *vanishing dimension* of $\alpha$ and $\beta$ is defined as the numbers

$$\operatorname{vdim}\alpha = \inf\left\{u\in\mathbb{Z}\;\middle|\; \begin{array}{l}\alpha\otimes\sigma=0 \text{ for all } \sigma\in\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)\\ \text{with } \dim\sigma<\operatorname{codim}\mathfrak{X}-u\end{array}\right\} \text{ and}$$

$$\operatorname{vdim}\beta = \inf\left\{v\in\mathbb{Z}\;\middle|\; \begin{array}{l}\operatorname{Hom}(\sigma,\beta)=0 \text{ for all } \sigma\in\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)\\ \text{with } \dim\sigma<\operatorname{codim}\mathfrak{X}-v\end{array}\right\}.$$

In particular, the vanishing dimension of an element in a Grothendieck space is $-\infty$ if and only if the element is trivial, and the vanishing dimension is less than or equal to 0 if and only if the element satisfies vanishing.

5.2. **Remark.** If $X$ is a complex in $\mathsf{P}^{\mathsf{f}}(R)$ with $\mathfrak{X}=\operatorname{Supp}X$, then the element $\alpha=[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ satisfies vanishing exactly when

$$\chi(X,Y)=0 \quad \text{for all complexes } Y\in\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp}Y)<\operatorname{codim}\mathfrak{X},$$

and $\alpha$ satisfies weak vanishing exactly when

$$\chi(X,Y)=0 \quad \text{for all complexes } Y\in\mathsf{P}^{\mathsf{f}}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp}Y)<\operatorname{codim}\mathfrak{X}.$$

The vanishing dimension of $\alpha$ measures the extent to which vanishing fails to hold: the vanishing dimension of $\alpha$ is the infimum of integers $u$ such that

$$\chi(X,Y)=0 \quad \text{for all complexes } Y\in\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp}Y)<\operatorname{codim}\mathfrak{X}-u.$$

It follows that the ring $R$ satisfies vanishing (or weak vanishing, respectively) as defined in 3.1, if and only if all elements of $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec}R$ satisfy vanishing (or weak vanishing, respectively).

If $Y$ is a complex in $\mathsf{I}^{\mathsf{f}}(R)$ with $\mathfrak{X}=\operatorname{Supp}Y$, then the element $\beta=[Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ satisfies vanishing exactly when

$$\xi(X,Y)=0 \quad \text{for all complexes } X\in\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp}X)<\operatorname{codim}\mathfrak{X}.$$

and $\beta$ satisfies weak vanishing exactly when

$$\xi(X,Y)=0 \quad \text{for all complexes } X\in\mathsf{P}^{\mathsf{f}}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp}X)<\operatorname{codim}\mathfrak{X}.$$

The vanishing dimension of $\beta$ measures the extent to which vanishing of the Euler form fails to hold: the vanishing dimension of $\beta$ is the infimum of integers $v$ such that

$$\xi(X,Y)=0 \quad \text{for all complexes } X\in\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp}X)<\operatorname{codim}\mathfrak{X}-v.$$

Because of the formulas in (3.2.1), it follows that the ring $R$ satisfies vanishing (or weak vanishing, respectively) if and only all elements of $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec}R$ satisfy vanishing (or weak vanishing, respectively).

5.3. **Remark.** For a specialization closed subset $\mathfrak{X}$ of $\operatorname{Spec}R$ and elements $\alpha\in\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, $\beta\in\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ and $\gamma\in\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, we have the following formulas for dimension.

$$\begin{aligned}\dim\gamma &= \dim\gamma^{\dagger},\\ \dim\alpha &= \dim\alpha^{\dagger}=\dim\alpha^{*}=\dim(D\otimes\alpha) \text{ and}\\ \dim\beta &= \dim\beta^{\dagger}=\dim\beta^{\star}=\dim\operatorname{Hom}(D,\beta).\end{aligned}$$

These follow immediately from the fact that the dagger, star and Foxby functors do not change supports of complexes. Further, we have the following formulas for vanishing dimension.

$$\operatorname{vdim}\alpha = \operatorname{vdim}\alpha^{\dagger} = \operatorname{vdim}\alpha^{*} = \operatorname{vdim}(D \otimes \alpha) \text{ and}$$
$$\operatorname{vdim}\beta = \operatorname{vdim}\beta^{\dagger} = \operatorname{vdim}\beta^{\star} = \operatorname{vdim}\operatorname{Hom}(D, \beta).$$

These follow immediately from the above together with (3.2.1).

5.4. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let* $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ *and let* $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$*. Then the following hold.*

(*i*) *If* $\operatorname{codim}\mathfrak{X} \leqslant 2$ *then vanishing holds for all elements in* $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ *and* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$*. In particular, we always have*

$$\operatorname{vdim}\alpha, \operatorname{vdim}\beta \leqslant \max(0, \operatorname{codim}\mathfrak{X} - 2).$$

(*ii*) *Let $\mathfrak{X}'$ be a specialization-closed subset of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$*. Then*

$$\operatorname{vdim}\overline{\alpha} \leqslant \operatorname{vdim}\alpha - (\operatorname{codim}\mathfrak{X} - \operatorname{codim}\mathfrak{X}') \textit{ and}$$

*for* $\overline{\alpha} \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}')$*. For any given $s$ in the range* $0 \leqslant s \leqslant \operatorname{vdim}\alpha$*, we can always find an $\mathfrak{X}'$ with* $s = \operatorname{codim}\mathfrak{X} - \operatorname{codim}\mathfrak{X}'$ *such that the above inequality becomes an equality. Likewise,*

$$\operatorname{vdim}\overline{\beta} \leqslant \operatorname{vdim}\beta - (\operatorname{codim}\mathfrak{X} - \operatorname{codim}\mathfrak{X}')$$

*for* $\overline{\beta} \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}')$*, and for any given $s$ in the range* $0 \leqslant s \leqslant \operatorname{vdim}\beta$*, we can always find an $\mathfrak{X}'$ with* $s = \operatorname{codim}\mathfrak{X} - \operatorname{codim}\mathfrak{X}'$ *such that the above inequality becomes an equality.*

(*iii*) *The element $\alpha$ satisfies weak vanishing if and only if, for all specialization-closed subsets $\mathfrak{X}'$ with $\mathfrak{X} \subseteq \mathfrak{X}'$ and* $\operatorname{codim}\mathfrak{X}' = \operatorname{codim}\mathfrak{X} - 1$,

$$\overline{\alpha} = 0 \quad \textit{as an element of} \quad \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}').$$

*Similarly, the element $\beta$ satisfies weak vanishing if and only if, for all specialization-closed subsets $\mathfrak{X}'$ with $\mathfrak{X} \subseteq \mathfrak{X}'$ and* $\operatorname{codim}\mathfrak{X}' = \operatorname{codim}\mathfrak{X} - 1$,

$$\overline{\beta} = 0 \quad \textit{as an element of} \quad \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}').$$

*Proof.* Because of Proposition 4.9 and the formulas in Remark 5.3, it suffices to consider the statements for $\alpha$ and $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. But the in this case, (*i*) and (*ii*) are already contained in [11, Example 6 and Remark 7], and (*iii*) follows by considerations similar to those proving (*ii*) in [11, Remark 7]. □

The following two propositions present conditions that are equivalent to having a certain vanishing dimension for elements of the Grothendieck space $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. There are similar results for elements of the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$; see [11, Proposition 23 and 24].

5.5. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, and let* $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$*. Then the following conditions are equivalent.*

(*i*) $\operatorname{vdim}\beta \leqslant 0$.
(*ii*) $\operatorname{Hom}(\gamma, \beta) = 0$ *for all* $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^{c})$ *with* $\dim\gamma < \operatorname{codim}\mathfrak{X}$.
(*iii*) $\overline{\beta} = 0$ *in* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}')$ *for any specialization-closed subset $\mathfrak{X}'$ of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim}\mathfrak{X}' < \operatorname{codim}\mathfrak{X}$.
(*iv*) $\overline{\beta} = 0$ *in* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}')$ *for any specialization-closed subset $\mathfrak{X}'$ of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim}\mathfrak{X}' = \operatorname{codim}\mathfrak{X} - 1$.

*Proof.* By definition $(i)$ is equivalent to $(ii)$, and Proposition 4.3 in conjunction with Remark 5.2 shows that $(i)$ implies $(iii)$. Clearly $(iii)$ is stronger than $(iv)$, and $(iv)$ in conjunction with Proposition 5.4 implies $(ii)$. □

5.6. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, and let $u$ be a non-negative integer. Then the following conditions are equivalent.*

- $(i)$ $\operatorname{vdim} \beta \leqslant v$.
- $(ii)$ $\operatorname{Hom}(\gamma, \beta) = 0$ *for all* $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$ *with* $\dim \gamma < \operatorname{codim} \mathfrak{X} - v$.
- $(iii)$ $\overline{\beta} = 0$ *in of* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim} \mathfrak{X}' < \operatorname{codim} \mathfrak{X} - u$.
- $(iv)$ $\overline{\beta} = 0$ *in* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim} \mathfrak{X}' = \operatorname{codim} \mathfrak{X} - v - 1$.

*Proof.* The structure of the proof is similar to that of Proposition (5.5). □

## 6. Grothendieck spaces in prime characteristic

According to [11, Theorem 19] the endomorphism $\Phi_{\mathfrak{X}}$ on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is diagonalizable; the precise statement is recalled in the next theorem. This section establishes that the endomorphism $\Psi_{\mathfrak{X}}$ on $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ is also diagonalizable; the precise statement is Theorem 6.2 below.

6.1. **Theorem.** *Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*. If $\alpha$ is an element in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $u$ is a non-negative integer with $u \geqslant \operatorname{vdim} \alpha$, then*

$$(p^u \Phi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id}) \circ (\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0,$$

*and there exists a unique decomposition*

$$\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$$

*in which each $\alpha^{(i)}$ is either zero or an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $p^{-i}$. The elements $\alpha^{(i)}$ can be computed according to the formula*

$$\begin{pmatrix} \alpha^{(0)} \\ \vdots \\ \alpha^{(u)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & p^{-1} & \cdots & p^{-u} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p^{-u} & \cdots & p^{-u^2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \Phi_{\mathfrak{X}}(\alpha) \\ \vdots \\ \Phi_{\mathfrak{X}}^u(\alpha) \end{pmatrix},$$

*and may also be recursively obtained as*

$$\alpha^{(0)} = \lim_{e \to \infty} \Phi_{\mathfrak{X}}^e(\alpha) \quad \textit{and} \quad \alpha^{(i)} = \lim_{e \to \infty} p^{ie} \Phi_{\mathfrak{X}}^e(\alpha - (\alpha^{(0)} + \cdots + \alpha^{(i-1)})).$$

6.2. **Theorem.** *Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*. If $\beta$ is an element in $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $v$ is a non-negative integer with $v \geqslant \operatorname{vdim} \beta$, then*

$$(p^v \Psi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Psi_{\mathfrak{X}} - \mathrm{id}) \circ (\Psi_{\mathfrak{X}} - \mathrm{id})(\beta) = 0,$$

*and there exists a unique decomposition*

$$\beta = \beta^{(0)} + \cdots + \beta^{(v)},$$

*in which each $\beta^{(i)}$ is either zero or an eigenvector for $\Psi_{\mathfrak{X}}$ with eigenvalue $p^{-i}$. The elements $\beta^{(i)}$ can be computed according to the formula*

$$\begin{pmatrix}\beta^{(0)}\\ \vdots \\ \beta^{(u)}\end{pmatrix} = \begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & p^{-1} & \cdots & p^{-v}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & p^{-v} & \cdots & p^{-v^2}\end{pmatrix}^{-1}\begin{pmatrix}\beta\\ \Psi_{\mathfrak{X}}(\beta)\\ \vdots\\ \Psi_{\mathfrak{X}}^{v}(\beta)\end{pmatrix}, \tag{6.2.1}$$

*and may also be recursively obtained as*

$$\beta^{(0)} = \lim_{e\to\infty}\Psi_{\mathfrak{X}}^{e}(\beta) \quad \textit{and} \quad \beta^{(i)} = \lim_{e\to\infty} p^{ie}\Psi_{\mathfrak{X}}^{e}(\beta - (\beta^{(0)} + \cdots + \beta^{(i-1)})).$$

*Proof.* On the injective Grothendieck space $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, the identities described in Lemma 2.11 imply that we have the following commutative diagram.

$$\begin{array}{ccc}
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) & \xrightarrow[\cong]{\Phi_{\mathfrak{X}}^{n}} & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})\\
{\scriptstyle (-)^{\dagger}}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\uparrow{\scriptstyle (-)^{\dagger}}\\
\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) & \xrightarrow{\Psi_{\mathfrak{X}}^{n}} & \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})
\end{array}$$

In particular,

$$\Psi_{\mathfrak{X}}(-) = (-)^{\dagger}\circ\Phi_{\mathfrak{X}}\circ(-)^{\dagger}.$$

By Remark 5.3, we have $v \geqslant \operatorname{vdim}\beta = \operatorname{vdim}\beta^{\dagger}$, so Theorem 6.1 and the above identity yields that

$$(p^{v}\Psi_{\mathfrak{X}} - \mathrm{id})\circ\cdots\circ(p\Psi_{\mathfrak{X}} - \mathrm{id})\circ(\Psi_{\mathfrak{X}} - \mathrm{id})(\beta) = 0. \tag{6.2.2}$$

Applying $\Psi_{\mathfrak{X}}^{e-v}$ to (6.2.2) results in a recursive formula to compute $\Psi_{\mathfrak{X}}^{e+1}(\beta)$ from $\Psi_{\mathfrak{X}}^{e}(\beta),\dots,\Psi_{\mathfrak{X}}^{e-v}(\beta)$. The characteristic polynomial for the recursion is

$$(p^{v}x-1)\cdots(px-1)(x-1),$$

which has $v+1$ distinct roots $1, p^{-1},\dots,p^{-v}$. Consequently, there exist elements $\beta^{(0)},\dots,\beta^{(v)}$ such that

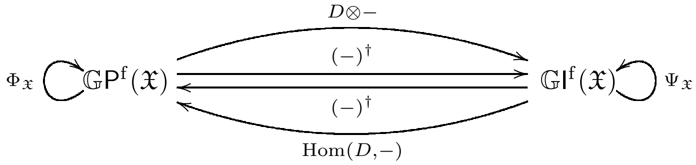
$$\Psi_{\mathfrak{X}}^{e}(\beta) = \beta^{(0)} + p^{-e}\beta^{(1)} + \cdots + p^{-ve}\beta^{(v)},$$

where each $\beta^{(i)}$ is an eigenvector for $\Psi_{\mathfrak{X}}$ with eigenvalue $p^{-i}$. Setting $e = 0$ obtains the decomposition $\beta = \beta^{(0)} + \cdots + \beta^{(v)}$, and solving the system of linear equations obtained by setting $e = 0,\dots,v$ shows (6.2.1); observe that the matrix is the Vandermonde matrix on $1, p^{-1},\dots,p^{-v}$, which is invertible. The formula also immediately shows that $\lim_{e\to\infty}\Psi^{e}(\beta) = \beta^{(0)}$ and that

$$\begin{aligned}
\lim_{e\to\infty} p^{ie}\Psi_{\mathfrak{X}}^{e}(\beta - (\beta^{(0)} + \cdots + \beta^{(i-1)})) &= \lim_{e\to\infty} p^{ie}\Psi_{\mathfrak{X}}^{e}(\beta^{(i)} + \cdots + \beta^{(v)})\\
&= \lim_{e\to\infty}(\beta^{(i)} + \cdots + p^{-(v-i)e}\beta^{(v)})\\
&= \beta^{(i)}.
\end{aligned}$$

This concludes the argument. $\square$

6.3. **Proposition.** *Assume that $R$ is a complete ring of prime characteristic $p$ and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$. *Consider the following diagram.*




*For the Grothendieck space $\mathbb{G}\mathsf{I}^\mathrm{f}(\mathfrak{X})$, we have the following identities.*

$$\Psi_\mathfrak{X}(-) = \Phi_\mathfrak{X}(-^\dagger)^\dagger = D \otimes \Phi_\mathfrak{X}(\operatorname{Hom}(D,-)).$$
$$(-)^{(i)} = (-)^{\dagger(i)\dagger} = D \otimes (\operatorname{Hom}(D,-)^{(i)}).$$

*Proof.* The formulas in the first line are an immediate consequence of Lemma 2.11. Let $\beta$ be an element in $\mathbb{G}\mathsf{I}^\mathrm{f}(\mathfrak{X})$. Using the decomposition in $\mathbb{G}\mathsf{P}^\mathrm{f}(\mathfrak{X})$ from Theorem 6.1, we can write

$$\beta = \beta^{\dagger\dagger} = \beta^{\dagger(0)\dagger} + \cdots + \beta^{\dagger(v)\dagger},$$

and since

$$\Psi_\mathfrak{X}(\beta^{\dagger(i)\dagger}) = \Phi_\mathfrak{X}(\beta^{\dagger(i)})^\dagger = p^{-i}\beta^{\dagger(i)\dagger},$$

we learn from the uniqueness of the decomposition that $\beta^{(i)} = \beta^{\dagger(i)\dagger}$. This proves the first equality in the second line. The last equality follows by similar considerations. □

6.4. **Remark.** In [11, Remark 21] it is established that the Dutta multiplicity is computable. Employing Theorems 6.1 and 6.2 together with the fact from Proposition 4.7 that the induced Hom-homomorphism on Grothendieck spaces is continuous in both variables, it follows, as will be shown below, that the two analogs of Dutta multiplicity are also computable.

Let $X$ and $Y$ be finite complexes. Set $\mathfrak{X} = \operatorname{Supp} X$ and $\mathfrak{Y} = \operatorname{Supp} Y$, and assume that $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$. Then, in the case where $Y$ is in $\mathsf{I}^\mathrm{f}(R)$, the multiplicity $\xi_\infty(X,Y)$ can be identified via (4.4.1) with the element

$$\begin{aligned}\lim_{e\to\infty} \operatorname{Hom}([X]_{\mathsf{D}^\mathrm{f}_\square(\mathfrak{Y}^c)}, \Psi^e_\mathfrak{Y}(Y)) &= \operatorname{Hom}([X]_{\mathsf{D}^\mathrm{f}_\square(\mathfrak{Y}^c)}, \lim_{e\to\infty} \Psi^e_\mathfrak{Y}(Y)) \\ &= \operatorname{Hom}([X]_{\mathsf{D}^\mathrm{f}_\square(\mathfrak{Y}^c)}, [Y]^{(0)}_{\mathsf{I}^\mathrm{f}(\mathfrak{Y})}),\end{aligned}$$

whereas, in the case where $X$ is in $\mathsf{P}^\mathrm{f}(R)$, the multiplicity $\xi^\infty(X,Y)$ can be identified via (4.4.1) with the element

$$\begin{aligned}\lim_{e\to\infty} \operatorname{Hom}(\Phi^e_\mathfrak{X}(X), [Y]_{\mathsf{D}^\mathrm{f}_\square(\mathfrak{X}^c)}) &= \operatorname{Hom}(\lim_{e\to\infty} \Phi^e_\mathfrak{X}(X), [Y]_{\mathsf{D}^\mathrm{f}_\square(\mathfrak{X}^c)}) \\ &= \operatorname{Hom}([X]^{(0)}_{\mathsf{P}^\mathrm{f}(\mathfrak{X})}, [Y]_{\mathsf{D}^\mathrm{f}_\square(\mathfrak{X}^c)}).\end{aligned}$$

The formulas in Theorems 6.1 and 6.2 now yield formulas for $\xi_\infty(X,Y)$ and $\xi^\infty(X,Y)$ as presented in the corollary below.

6.5. **Corollary.** *Assume that $R$ is a complete ring of prime characteristic $p$ and with perfect residue field. Let $X$ and $Y$ be finite complexes with*

$$\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\} \quad \textit{and} \quad \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R.$$

*When $Y \in \mathsf{I}^{\mathrm{f}}(R)$, letting $v$ denote the vanishing dimension of $[Y]_{\mathsf{I}^{\mathrm{f}}(\operatorname{Supp} Y)}$ and setting $t = \operatorname{codim}(\operatorname{Supp} Y)$, we have*

$$\xi_\infty(X,Y) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ p^t & p^{t-1} & \cdots & p^{t-v} \\ \vdots & \vdots & \ddots & \vdots \\ p^{vt} & p^{v(t-1)} & \cdots & p^{v(t-v)} \end{pmatrix}^{-1} \begin{pmatrix} \xi(X,Y) \\ \xi(X,\mathbf{R}G(Y)) \\ \vdots \\ \xi(X,\mathbf{R}G^v(Y)) \end{pmatrix},$$

*and when $X \in \mathsf{P}^{\mathrm{f}}(R)$, letting $u$ denote the vanishing dimension of $[X]_{\mathsf{P}^{\mathrm{f}}(\operatorname{Supp} X)}$ and setting $s = \operatorname{codim}(\operatorname{Supp} X)$, we have*

$$\xi^\infty(X,Y) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ p^s & p^{s-1} & \cdots & p^{s-u} \\ \vdots & \vdots & \ddots & \vdots \\ p^{us} & p^{u(s-1)} & \cdots & p^{u(s-u)} \end{pmatrix}^{-1} \begin{pmatrix} \xi(X,Y) \\ \xi(\mathbf{L}F(X),Y) \\ \vdots \\ \xi(\mathbf{L}F^u(X),Y) \end{pmatrix}.$$

*Thus, it is possible to calculate $\xi_\infty(X,Y)$ and $\xi^\infty(X,Y)$ as $\mathbb{Q}$–linear combinations of ordinary Euler forms; in particular, they are rational numbers.*

Note that the above corollary also can be obtained directly from [11, Remark 21] by employing Lemma 2.11 and the formulas in (3.2.1).

6.6. **Remark.** Let $\mathfrak{X}$ and $\mathfrak{X}'$ be specialization-closed subsets of $\operatorname{Spec} R$ such that $\mathfrak{X} \subseteq \mathfrak{X}'$. Set $s = \operatorname{codim} \mathfrak{X} - \operatorname{codim} \mathfrak{X}'$ and consider the inclusion homomorphism

$$\overline{(-)}\colon \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}').$$

Pick an element $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, and apply the convention that $\beta^{(t)} = 0$ for all negative integers $t$. It follows immediately that

$$\Psi_{\mathfrak{X}'}(\overline{\beta}) = p^s \overline{\Psi_{\mathfrak{X}}(\beta)},$$

and employing Theorem 6.2 we obtain the identity $\overline{\beta^{(i)}} = \overline{\beta}^{(i-s)}$. The situation may be visualized as follows

$$\begin{array}{l} \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \ni \beta = \beta^{(0)} + \cdots + \beta^{(s)} + \beta^{(s+1)} + \cdots + \beta^{(v)} \\ \qquad\qquad\ \ \downarrow \\ \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}') \ni \overline{\beta} = \overline{\beta}^{(0)} + \overline{\beta}^{(1)} + \cdots + \overline{\beta}^{(v-s)}. \end{array}$$

There are similar results for elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$; see [11, Remark 20].

The following two propositions characterize vanishing dimension for elements of the Grothendieck space $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. They should be read in parallel with Propositions 5.5 and 5.6. There are similar results for the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$; see [11, Proposition 23 and 24].

6.7. **Proposition.** *Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$ *and let $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. The following are equivalent.*

(*i*) *$\beta$ satisfies vanishing.*
(*ii*) $\operatorname{vdim} \beta \leqslant 0$.
(*iii*) $\beta = \beta^{(0)}$.
(*iv*) $\beta = \Psi_{\mathfrak{X}}(\beta)$.

(*v*) $\beta = \Psi^e_{\mathfrak{X}}(\beta)$ *for some* $e \in \mathbb{N}$.
(*vi*) $\beta = \lim_{e\to\infty} \Psi^e_{\mathfrak{X}}(\beta)$.

*Proof.* By definition (*i*) and (*ii*) are equivalent, and from Theorem 6.2 it follows that (*ii*) implies (*iii*). Moreover, Theorem 6.2 shows that the four conditions (*iii*)–(*vi*) are equivalent. Finally, condition (*iii*) implies condition (*i*) through a reference to Remark 6.6 and Proposition 5.5. □

6.8. **Proposition.** *Assume that* $R$ *is complete of prime characteristic* $p$ *and with perfect residue field. Let* $\mathfrak{X}$ *be a specialization-closed subset of* $\operatorname{Spec} R$*, let* $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ *and let* $v$ *be a non-negative integer. The following are equivalent.*

(*i*) $\operatorname{vdim} \beta \leqslant v$.
(*ii*) $\beta = \beta^{(0)} + \cdots + \beta^{(v)}$.
(*iii*) $(p^v\Psi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Psi_{\mathfrak{X}} - \mathrm{id}) \circ (\Psi_{\mathfrak{X}} - \mathrm{id})(\beta) = 0$.

*Proof.* From Theorem 6.2 it follows that (*i*) implies (*ii*) which is equivalent to (*iii*). Since $\beta^{(i)} \neq 0$ implies $\operatorname{vdim} \beta^{(i)} = i$ by Remark 6.6 and Proposition 5.6, it follows that (*ii*) implies (*i*). □

6.9. **Proposition.** *Assume that* $R$ *is complete of prime characteristic* $p$ *and with perfect residue field. Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be specialization-closed subsets of* $\operatorname{Spec} R$ *such that* $\mathfrak{X}\cap\mathfrak{Y} = \{\mathfrak{m}\}$ *and* $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$*, and set* $e = \dim R - (\dim \mathfrak{X} + \dim \mathfrak{Y})$. *If* $(\sigma, \tau)$ *is a pair of elements from*

$$\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}), \quad \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}) \quad \textit{or} \quad \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),$$

*so that* $\sigma \otimes \tau$ *is a well-defined element of* $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ *or* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$*, then*

$$(\sigma \otimes \tau)^{(i)} = \sum_{m+n=i+e} \sigma^{(m)} \otimes \tau^{(n)}. \tag{6.9.1}$$

*If instead* $(\sigma, \tau)$ *is a pair of elements from*

$$\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}), \quad \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}) \quad \textit{or} \quad \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}),$$

*so that* $\operatorname{Hom}(\sigma, \tau)$ *is a well-defined element of* $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ *or* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$*, then*

$$\operatorname{Hom}(\sigma, \tau)^{(i)} = \sum_{m+n=i+e} \operatorname{Hom}(\sigma^{(m)}, \tau^{(n)}).$$

*Proof.* We will verify that (6.9.1) holds in the case where $(\sigma, \tau)$ is pair of elements from $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y})$. The verification of the remaining statements is similar.

It suffices to argue that the element

$$\alpha = \sum_{m+n=i+e} \sigma^{(m)} \otimes \tau^{(n)} \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$$

is an eigenvector for $\Phi_{\mathfrak{m}} = \Phi_{\mathfrak{X}\cap\mathfrak{Y}}$ with eigenvalue $p^{-i}$. We compute

$$\begin{aligned}\Phi_{\mathfrak{m}}(\alpha) &= \sum_{m+n=i+e} p^{-\dim R} F_{\mathfrak{m}}(\sigma^{(m)} \otimes \tau^{(n)}) \\ &= p^{-\dim R} \sum_{m+n=i+e} F_{\mathfrak{X}}(\sigma^{(m)}) \otimes F_{\mathfrak{Y}}(\tau^{(n)}) \\ &= p^{-\dim R} \sum_{m+n=i+e} p^{\operatorname{codim}\mathfrak{X}} \Phi_{\mathfrak{X}}(\sigma^{(m)}) \otimes p^{\operatorname{codim}\mathfrak{Y}} \Phi_{\mathfrak{Y}}(\tau^{(n)}) \\ &= p^{-i} \sum_{m+n=i+e} \sigma^{(m)} \otimes \tau^{(n)} = p^{-i}\alpha.\end{aligned}$$

Here, all equalities but the second are propelled only by definitions. The second equality follows from Proposition 2.13. □

In [11], the concept of "numerical vanishing" is introduced for elements $\alpha$ of the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$. We here repeat the definition and extend it to elements $\beta$ in the Grothendieck space $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$.

6.10. **Definition.** Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. An element $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is said to *satisfy numerical vanishing* if the images in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$ of $\alpha$ and $\alpha^{(0)}$ coincide. An element $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ is said to *satisfy numerical vanishing* if the images in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$ of $\beta$ and $\beta^{(0)}$ coincide. The ring $R$ is said to *satisfy numerical vanishing* if all elements of the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ satisfy numerical vanishing for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$.

6.11. **Remark.** $R$ satisfies numerical vanishing precisely when all elements of the Grothendieck space $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ satisfy numerical vanishing for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$. To see this, simply note that, by Proposition 6.3, the element $\beta$ in $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ satisfies numerical vanishing if and only if the corresponding element $\beta^{\dagger}$ in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ does. This observation allows us in the following proposition to present an injective version of [11, Remark 28].

6.12. **Proposition.** *Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. A necessary and sufficient condition for $R$ to satisfy numerical vanishing is that all element of $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$ satisfy numerical vanishing: that is, that*

$$\chi(\mathbf{R}G(Y)) = p^{\dim R}\chi(Y)$$

*for all complexes $Y \in \mathsf{I}^{\mathsf{f}}(\mathfrak{m})$. If $R$ is Cohen–Macaulay, $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$ is generated by modules, and hence a necessary and sufficient condition for $R$ to satisfy numerical vanishing is that*

$$\operatorname{length} G(N) = p^{\dim R} \operatorname{length} N$$

*for all modules $N$ with finite length and finite injective dimension.*

*Proof.* The proposition follows immediately from [11, Remark 28] by applying the dagger duality isomorphism between $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m})$ and $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$ and by noting that $(-)^{\dagger}$ takes a module in $\mathsf{P}^{\mathsf{f}}(\mathfrak{m})$ to a module in $\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$ and vice versa. □

## 7. Self-duality

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$, and let $K$ be a Koszul complex in $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ on $\operatorname{codim}\mathfrak{X}$ elements. It is a well-know fact that Koszul complexes are "self-dual" in the sense that $K \simeq \Sigma^{\operatorname{codim}\mathfrak{X}} K^*$. In particular, for the element $\alpha = [K]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, we have

$$\alpha = [\Sigma^{\operatorname{codim}\mathfrak{X}} K^*]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} = (-1)^{\operatorname{codim}\mathfrak{X}}[K^*]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} = (-1)^{\operatorname{codim}\mathfrak{X}}\alpha^*.$$

Proposition 7.4 below shows that this feature is displayed for all elements that satisfy vanishing.

7.1. **Definition.** Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and consider an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and an element $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$. If

$$\alpha = (-1)^{\operatorname{codim}\mathfrak{X}}\alpha^*,$$

we say that $\alpha$ is *self-dual*, and if all elements in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ are self-dual, we say that $R$ *satisfies self-duality.* Moreover, if the above equality holds after an application of the inclusion homomorphism $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ so that

$$\overline{\alpha} = (-1)^{\operatorname{codim}\mathfrak{X}}\overline{\alpha^*}$$

in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, we say that $\alpha$ is *numerically self-dual*, and if all elements in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ are numerically self-dual, we say that $R$ *satisfies numerical self-duality.*

Similarly, if

$$\beta = (-1)^{\operatorname{codim}\mathfrak{X}}\beta^\star,$$

we say that $\beta$ is *self-dual*, and if the above equality holds after an application of the inclusion homomorphism $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ so that

$$\overline{\beta} = (-1)^{\operatorname{codim}\mathfrak{X}}\overline{\beta^\star}$$

in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, we say that $\beta$ is *numerically self-dual.*

7.2. **Remark.** The commutativity of the star and dagger functors shows that an element $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ is self-dual if and only if the corresponding element $\beta^\dagger \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is self-dual. Thus, $R$ satisfies self-duality if and only if all elements in $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ are self-dual. A similar remark holds for numerical self-duality.

7.3. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and let $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$. Then,*

$$\operatorname{vdim}\alpha^* = \operatorname{vdim}\alpha \quad \textit{and} \quad \operatorname{vdim}\beta^\star = \operatorname{vdim}\beta.$$

*If, in addition, $R$ is complete of prime characteristic $p$ and with perfect residue field, we have*

$$\Phi_{\mathfrak{X}}(\alpha^*) = \Phi_{\mathfrak{X}}(\alpha)^* \quad \textit{and} \quad \Psi_{\mathfrak{X}}(\beta^\star) = \Psi_{\mathfrak{X}}(\beta)^\star.$$

*In particular, for all integers $i$, we have*

$$(\alpha^*)^{(i)} = (\alpha^{(i)})^* \quad \textit{and} \quad (\beta^\star)^{(i)} = (\beta^{(i)})^\star.$$

*Proof.* The formulas for vanishing dimension follow from (3.2.1), since the dagger functor does not change the dimension of a complex. The second pair of formulas follow immediately from the commutativity of the star and Frobenius functors; see 2.10. Thus, it follows that

$$\Phi_{\mathfrak{X}}(\alpha^{(i)*}) = \Phi_{\mathfrak{X}}(\alpha^{(i)})^* = p^{-i}\alpha^{(i)*}.$$

That is to say, $\alpha^{(i)*}$ is an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $p^{-i}$. Setting $u = \operatorname{vdim}\alpha$, the decomposition

$$\alpha^* = (\alpha^{(0)} + \cdots + \alpha^{(u)})^* = \alpha^{(0)*} + \cdots + \alpha^{(u)*}$$

now shows that $\alpha^{*(i)} = \alpha^{(i)*}$. A similar argument applies for $\beta$. □

7.4. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$. *If an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ satisfies vanishing, then $\alpha$ is self-dual, and if an element $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ satisfies vanishing, then $\beta$ is self-dual. Moreover, $R$ satisfies vanishing if and only if $R$ satisfies self-duality, and if $R$ satisfies numerical self-duality, then $R$ satisfies weak vanishing.*

*Proof.* We shall prove that, if $\alpha$ satisfies vanishing, then $\alpha$ is self-dual. The corresponding statement for $\beta$ follows from dagger duality, since $\beta$ is self-dual exactly when $\beta^\dagger$ is and satisfies vanishing exactly when $\beta^\dagger$ does.

By Proposition 4.3, it suffices to assume that $\alpha = [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}$ for a complex $X$ from $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$. We are required to establish the identity

$$\chi(X^* \otimes^{\mathbf{L}}_R -) = (-1)^{\operatorname{codim}\mathfrak{X}}\chi(X \otimes^{\mathbf{L}}_R -) \tag{7.4.1}$$

viewed as metafunctions on $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)$. First, we translate this question into showing that, if $R$ is a domain and $\mathfrak{X}$ equals $\mathfrak{m}$, then

$$\chi(X^*) = (-1)^{\dim R}\chi(X)$$

for all complexes $X$ in $\mathsf{P}^{\mathsf{f}}(\mathfrak{m})$ such that $[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{m})}$ satisfies vanishing.

**1°** By assumption, $\alpha$ satisfies vanishing, and Proposition 7.3 implies that so does $\alpha^*$. From Proposition 4.3 we see that, in order to show (7.4.1), it suffices to test with modules of the form $R/\mathfrak{p}$ for prime ideals $\mathfrak{p}$ from $\mathfrak{X}^c$ with $\dim R/\mathfrak{p} = \operatorname{codim}\mathfrak{X}$. Consider the following computation.

$$\begin{aligned} X^* \otimes^{\mathbf{L}}_R R/\mathfrak{p} &= \mathbf{R}\mathrm{Hom}_R(X,R) \otimes^{\mathbf{L}}_R R/\mathfrak{p} \\ &\simeq \mathbf{R}\mathrm{Hom}_R(X,R/\mathfrak{p}) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(X,\mathbf{R}\mathrm{Hom}_{R/\mathfrak{p}}(R/\mathfrak{p},R/\mathfrak{p})) \\ &\simeq \mathbf{R}\mathrm{Hom}_{R/\mathfrak{p}}(X \otimes^{\mathbf{L}}_R R/\mathfrak{p}, R/\mathfrak{p}). \end{aligned}$$

Here, the first isomorphism follows from (Tensor-eval); the second is trivial; and the third is due to (Adjoint). To keep notation simple, let

$$(-)^{*_{R/\mathfrak{p}}} = \mathbf{R}\mathrm{Hom}_{R/\mathfrak{p}}(-,R/\mathfrak{p}).$$

We are required to demonstrate that

$$\chi(X^* \otimes^{\mathbf{L}}_R R/\mathfrak{p}) = (-1)^{\dim R/\mathfrak{p}}\chi(X \otimes^{\mathbf{L}}_R R/\mathfrak{p}),$$

and since the Euler characteristics $\chi^R$ and $\chi^{R/\mathfrak{p}}$ are identical on all finite $R/\mathfrak{p}$–complexes with finite length homology, the computations above imply that we need to demonstrate that

$$\chi^{R/\mathfrak{p}}((X \otimes^{\mathbf{L}}_R R/\mathfrak{p})^{*_{R/\mathfrak{p}}}) = (-1)^{\dim R/\mathfrak{p}}\chi^{R/\mathfrak{p}}(X \otimes^{\mathbf{L}}_R R/\mathfrak{p}).$$

Having changed rings from $R$ to the domain $R/\mathfrak{p}$, we need to verify that the element $[X \otimes^{\mathbf{L}}_R R/\mathfrak{p}]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{m}/\mathfrak{p})}$ in the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m}/\mathfrak{p})$ over $R/\mathfrak{p}$ satisfies vanishing. But this follows from the fact that $\alpha = [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ satisfies vanishing, since

$$\chi^{R/\mathfrak{p}}((X \otimes^{\mathbf{L}}_R R/\mathfrak{p}) \otimes^{\mathbf{L}}_{R/\mathfrak{p}} R/\mathfrak{a}) = \chi^R(X \otimes^{\mathbf{L}}_R R/\mathfrak{a}) = 0.$$

for all ideals $\mathfrak{a} \in V(\mathfrak{p})$ with $\dim R/\mathfrak{a} < \dim R/\mathfrak{p} = \operatorname{codim} \mathfrak{X}$. Thus, it suffices to show that

$$\chi(X^*) = (-1)^{\dim R}\chi(X)$$

when $R$ is a domain, $\mathfrak{X}$ equals $\{\mathfrak{m}\}$ and $[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{m})}$ satisfies vanishing.

**2°** Without loss of generality, we may assume that $R$ is complete; in particular, we may assume that $R$ admits a normalized dualizing complex $D$. Letting $Y = R$ in (3.2.1) and applying Proposition 4.3, it follows that

$$\chi(X^*) = \chi(X^*, R) = \chi(X, D) = \chi(X, \mathrm{H}(D)).$$

According to 2.6 we may assume that the modules in the dualizing complex $D$ have the form

$$D_i = \bigoplus_{\dim R/\mathfrak{p}=i} E_R(R/\mathfrak{p}). \tag{7.4.2}$$

Let $d = \dim R$ and observe that, since $[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{m})}$ satisfies vanishing and

$$\dim \mathrm{H}_i(D) \leqslant \dim D_i < d \quad \text{for all} \quad i < d,$$

it follows that

$$\chi^R(X^*) = \chi^R(X, \Sigma^d \mathrm{H}_d(D)) = (-1)^d \chi^R(X, \mathrm{H}_d(D)).$$

Since $\mathrm{H}_d(D)$ is a submodule of $D_d$, there is a short exact sequence

$$0 \to \mathrm{H}_d(D) \to D_d \to Q \to 0, \tag{7.4.3}$$

where $Q$ is a submodule of $D_{d-1}$, so that $\dim Q \leqslant \dim D_{d-1} \leqslant d-1$, where the last inequality follows from (7.4.2). Since $R$ is assumed to be a domain,

$$D_d = E(R) = R_{(0)},$$

so localizing the short exact sequence (7.4.3) at the prime ideal (0), we obtain an isomorphism

$$\mathrm{H}_d(D)_{(0)} \xrightarrow{\cong} R_{(0)}.$$

This lifts to an $R$–homomorphism, producing an exact sequence of finitely generated $R$–modules

$$0 \to K \to \mathrm{H}_d(D) \to R \to C \to 0,$$

where $K$ and $C$ are not supported at the prime ideal (0). Thus, $\dim K$ and $\dim C$ are strictly smaller than $\dim R$. Consequently, since $[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ satisfies vanishing and the intersection multiplicity is additive on short exact sequences,

$$\begin{aligned}\chi(X^*) &= (-1)^d \chi(X, \mathrm{H}_d(D)) \\ &= (-1)^d(\chi(X, K) + \chi(X, R) - \chi(X, C)) \\ &= (-1)^d \chi(X),\end{aligned}$$

which concludes the argument.

**3°** We have now shown that, if $R$ satisfies vanishing, then $R$ satisfies self-duality. To see the other implication, assume that $R$ satisfies self-duality and let $\alpha$ be an element of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ for some specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$. For any specialization-closed subset $\mathfrak{X}'$ of $\operatorname{Spec} R$ with $\mathfrak{X} \subseteq \mathfrak{X}'$ and $\operatorname{codim} \mathfrak{X}' = \operatorname{codim} \mathfrak{X} - 1$, we now have, for the image $\overline{\alpha}$ of $\alpha$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}')$, that

$$(-1)^{\operatorname{codim} \mathfrak{X}'} \overline{\alpha}^* = \overline{\alpha} = \overline{(-1)^{\operatorname{codim} \mathfrak{X}} \alpha^*} = (-1)^{\operatorname{codim} \mathfrak{X}} \overline{\alpha}^*,$$

which means that $\overline{\alpha} = 0$. Thus, by Proposition 5.5, $\alpha$ satisfies vanishing, and since $\alpha$ was arbitrary, $R$ must satisfy vanishing. Considering instead the image $\overline{\alpha}$ of $\alpha$ in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}')$ and applying Proposition 5.4, the same argument shows that, if $\alpha$ is numerically self-dual, then $\alpha$ satisfies weak vanishing. Thus, if $R$ satisfies numerical self-duality, then $R$ satisfies weak vanishing. □

7.5. **Theorem.** *Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and let $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. Then, for all non-negative integers $i$*

$$(\alpha^*)^{(i)} = (-1)^{i+\operatorname{codim} \mathfrak{X}} \alpha^{(i)} \quad \textit{and} \quad (\beta^\star)^{(i)} = (-1)^{i+\operatorname{codim} \mathfrak{X}} \beta^{(i)}. \tag{7.5.1}$$

*Consequently, if $u$ is the vanishing dimension of $\alpha$, then*

$$(-1)^{\operatorname{codim} \mathfrak{X}} \alpha^* = \alpha^{(0)} - \alpha^{(1)} + \alpha^{(2)} - \cdots + (-1)^u \alpha^{(u)},$$

*and if $v$ is the vanishing dimension of $\beta$, then*

$$(-1)^{\operatorname{codim} \mathfrak{X}} \beta^\star = \beta^{(0)} - \beta^{(1)} + \beta^{(2)} - \cdots + (-1)^v \beta^{(v)}.$$

*Proof.* The last two statements of the proposition are immediate consequences of (7.5.1). We shall prove the formula for $\alpha$ in (7.5.1); the proof of the formula for $\beta$ is similar.

The proof is by induction on $i$. For $i = 0$, since $\alpha^{(0)}$ satisfies vanishing, the statement follows from Propositions 7.3 and 7.4, since

$$(\alpha^*)^{(0)} = (\alpha^{(0)})^* = (-1)^{\operatorname{codim} \mathfrak{X}} \alpha^{(0)}.$$

Next, assume that $i > 0$ and that the statement holds for smaller values of $i$. Choose an arbitrary specialization-closed subset $\mathfrak{X}'$ of $\operatorname{Spec} R$ such that $\mathfrak{X} \subseteq \mathfrak{X}'$ and $\operatorname{codim} \mathfrak{X}' = \operatorname{codim} \mathfrak{X} - 1$, and consider the element

$$\sigma = (\alpha^*)^{(i)} - (-1)^{\operatorname{codim} \mathfrak{X} + i} \alpha^{(i)}.$$

We want to show that $\sigma = 0$. Applying the automorphism $\Phi_{\mathfrak{X}}$, we get by Proposition 7.3 that

$$\begin{aligned} \Phi_{\mathfrak{X}}(\sigma) &= \Phi_{\mathfrak{X}}((\alpha^*)^{(i)}) - (-1)^{\operatorname{codim} \mathfrak{X} + i} \Phi_{\mathfrak{X}}(\alpha^{(i)}) \\ &= p^{-i}((\alpha^*)^{(i)} - (-1)^{\operatorname{codim} \mathfrak{X} + i} \alpha^{(i)}) = p^{-i} \sigma, \end{aligned}$$

showing that $\sigma$ is an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $p^{-i}$; in particular, we have $\sigma = \sigma^{(i)}$. Denote by $\bar{\sigma}$ the image of $\sigma$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}')$. Then, by [11, Remark 20] (which corresponds to Remark 6.6 but for elements of $\mathbb{G}(\mathfrak{X})$) and the induction hypothesis we obtain

$$\begin{aligned} \overline{\sigma} &= \overline{(\alpha^*)^{(i)}} - (-1)^{\operatorname{codim} \mathfrak{X} + i} \overline{\alpha^{(i)}} \\ &= \overline{\alpha}^{*(i-1)} - (-1)^{\operatorname{codim} \mathfrak{X}' + (i-1)} \overline{\alpha}^{(i-1)} = 0. \end{aligned}$$

Consequently, by [11, Proposition 23] (which corresponds to Proposition 5.5 but for elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$), $\sigma$ must satisfy vanishing: that is, $\sigma = \sigma^{(0)}$. But then $\sigma^{(i)} = \sigma = \sigma^{(0)}$ forcing $\sigma = 0$. □

7.6. **Remark.** Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and consider an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. In view of Theorem 6.1 we may decompose $\alpha$ into eigenvectors

$$\alpha = \alpha^{(0)} + \alpha^{(1)} + \alpha^{(2)} + \cdots + \alpha^{(u)}$$

where $u$ is the vanishing dimension of $\alpha$. Comparing it with the decomposition of $\alpha^*$ from Theorem 7.5

$$(-1)^{\operatorname{codim}\mathfrak{X}}\alpha^* = \alpha^{(0)} - \alpha^{(1)} + \alpha^{(2)} - \cdots + (-1)^u\alpha^{(u)}$$

shows that $\alpha$ is self-dual if and only if $\alpha^{(i)} = 0$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ for all odd $i$: that is, if and only if

$$\alpha = \alpha^{(0)} + \alpha^{(2)} + \cdots$$

in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. Similarly, $\alpha$ is numerically self-dual if and only if $\overline{\alpha^{(i)}} = 0$ in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ for all odd $i$: that is, if and only if

$$\overline{\alpha} = \overline{\alpha^{(0)}} + \overline{\alpha^{(2)}} + \cdots$$

in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. Similar considerations apply for elements $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$.

In Proposition 7.4, we proved that vanishing and self-duality are equivalent for $R$ and that numerical self-duality implies weak vanishing. The following proposition shows that, in characteristic $p$, numerical vanishing logically lies between self-duality and numerical self-duality.

7.7. **Proposition.** *Assume that $R$ is complete of prime characteristic $p$ and with perfect residue field. For the following conditions, each condition implies the next. In fact, (i) and (ii) are equivalent.*

(*i*) *$R$ satisfies vanishing.*
(*ii*) *$R$ satisfies self-duality.*
(*iii*) *$R$ satisfies numerical vanishing.*
(*iv*) *$R$ satisfies numerical self-duality.*
(*v*) *$R$ satisfies weak vanishing.*

*Proof.* The equivalence of (*i*) and (*ii*) and the fact that (*iv*) implies (*v*) is contained in Proposition 7.4. The fact that (*i*) implies (*iii*) is contained in [11, Proposition 27], and Remark 7.6 makes it clear that (*iii*) implies (*iv*). □

7.8. **Remark.** The constructions by Miller and Singh [15] shows that there can exist elements satisfying self-duality but not vanishing as well as elements satisfying numerical self-duality but not numerical vanishing; see [11, Example 35] for further details on this example. Roberts [20] has shown the existence of a ring satisfying weak vanishing but not numerical self-duality; see [11, Example 32] for further details. Thus, all the implications except the equivalence in the preceding proposition are strict.

7.9. **Proposition.** *$R$ satisfies vanishing precisely when*

$$\alpha \otimes \gamma = (-1)^{\operatorname{codim}\mathfrak{X}} \operatorname{Hom}(\alpha, \gamma) \tag{7.9.1}$$

*in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$, and $R$ satisfies numerical self-duality precisely when (7.9.1) holds in*

$\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ *when requiring* $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$ *instead. In other words, $R$ satisfies vanishing precisely when the intersection multiplicity and the Euler form satisfy the identity*

$$\chi(X,Y) = (-1)^{\operatorname{codim}(\operatorname{Supp} X)}\xi(X,Y) \tag{7.9.2}$$

*for all complexes* $X \in \mathsf{P}^{\mathrm{f}}(R)$ *and* $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ *with*

$$\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\} \quad \textit{and} \quad \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R,$$

*and $R$ satisfies numerical self-duality precisely when* (7.9.2) *holds when restricting to complexes* $Y \in \mathsf{P}^{\mathrm{f}}(R)$*.*

*Proof.* Employing Proposition 4.9 it is readily verified that (7.9.1) is equivalent to

$$\alpha \otimes \gamma = (-1)^{\operatorname{codim}\mathfrak{X}}\alpha^* \otimes \gamma.$$

However, this identity is satisfied for all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$ precisely when $\alpha$ is self-dual. From Proposition 7.4 it follows that $R$ satisfies vanishing if and only if (7.9.1) holds for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$.

On the other hand, applying the above argument to the case where $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$ shows that $R$ satisfies numerical self-duality presily when (7.9.1) is satisfied for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$.

Assume next that (7.9.1) holds in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$. If $X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ are complexes such that

$$\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\} \quad \text{and} \quad \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R, \tag{7.9.3}$$

the identity (7.9.2) follows by setting

$$\mathfrak{X} = \operatorname{Supp} X, \quad \alpha = [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \quad \text{and} \quad \gamma = [Y]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)}$$

in (7.9.1). Conversely, if (7.9.2) holds for all complexes $X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ such that (7.9.3) is satisfied, then (7.9.1) follows for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$, since we by Proposition 4.3, $\alpha = r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ for an $r \in \mathbb{Q}$ and a complex $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ with $\operatorname{codim}(\operatorname{Supp} X) = \operatorname{codim}\mathfrak{X}$. Applying the same argument to elements $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$ and complexes $Y \in \mathsf{P}^{\mathrm{f}}(R)$ proves the last part of the proposition. □

7.10. **Remark.** Proposition 7.9 confirms Chan's supposition in [4], in the setting of complexes rather than modules, that the formula in (7.9.2) is equivalent to the vanishing conjecture. Note that, when restricting attention to complexes $Y$ in $\mathsf{P}^{\mathrm{f}}(R)$, formula (7.9.2) is equivalent to numerical self-duality, which implies the weak vanishing conjecture but need not be equivalent to it. This negatively answers the question of whether the restriction of the formula in (7.9.2) to complexes $Y$ in $\mathsf{P}^{\mathrm{f}}(R)$ is equivalent to the weak vanishing conjecture.

We already know that, if $R$ is regular, then $R$ satisfies vanishing, whereas, if $R$ is a complete intersection (which is complete of prime characteristic $p$ and with perfect residue field), then $R$ satisfies numerical vanishing; see [11, Example 33]. The authors believe that this line of implications can be continued, at least in the characteristic $p$ case, with the claim that, if $R$ is Gorenstein, $R$ satisfies numerical self-duality, so that we have the following implications of properties of $R$ in the case where $R$ is complete of prime characteristic $p$ and with perfect residue field.

$$\begin{array}{ccccc}
\text{regular} & \Longrightarrow & \text{vanishing} & \Longleftrightarrow & \text{self-duality} \\
\Downarrow & & \Downarrow & & \\
\text{complete intersection} & \Longrightarrow & \text{numerical vanishing} & & \\
\Downarrow & & \Downarrow & & \\
\text{Gorenstein} & = = = = \Rightarrow & \text{numerical self-duality} & &
\end{array}$$

This supposition complies with the following proposition.

7.11. **Proposition.** *Assume that $R$ is Gorenstein and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$. *If* $\dim \mathfrak{X} \leqslant 2$, *then all elements of* $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ *are numerically self-dual. In particular, if* $\dim R \leqslant 5$, *then $R$ satisfies numerical self-duality.*

*Proof.* Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ with $\dim \mathfrak{X} \leqslant 2$ and consider elements $\alpha$ in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\beta$ in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}^c)$. Then $\operatorname{codim} \mathfrak{X}^c \leqslant 2$, and therefore $\beta$ satisfies vanishing by Proposition 5.4; in particular,

$$\beta^* = (-1)^{\operatorname{codim} \mathfrak{X}^c} \beta = (-1)^{\dim R - \operatorname{codim} \mathfrak{X}} \beta.$$

When $R$ is Gorenstein, the complex $D = \Sigma^{\dim R} R$ is a normalized dualizing complex for $R$ forcing $(-)^\dagger = \Sigma^{\dim R}(-)^*$. Thus, applying Proposition 4.9 the identity

$$\alpha^* \otimes \beta = \alpha \otimes \beta^\dagger = (-1)^{\dim R} \alpha \otimes \beta^* = (-1)^{\operatorname{codim} \mathfrak{X}} \alpha \otimes \beta$$

holds in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$. This proves that $\alpha^* = (-1)^{\operatorname{codim} \mathfrak{X}} \alpha$ so that $\alpha$ is numerically self-dual.

If $\dim R \leqslant 5$ then any specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$ must either satisfy $\operatorname{codim} \mathfrak{X} \leqslant 2$, in which case vanishing holds in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ by Proposition 5.4, or $\dim \mathfrak{X} \leqslant 2$. In either case, all elements of $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ are numerically self-dual. □

Since numerical self-duality implies weak vanishing, the preceding proposition shows that weak vanishing holds for any Gorenstein ring of dimension at most 5. Dutta [7] has already proven this fact.

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